The Simulation of Flood Planning Based on PDE

Yifeng Zhu, Jiakai Wang and Zhemin Liang

Summary

In this paper, the problem of flood caused by the breach of Saluda dam is addressed. The system of Saint Venant partial differential equations is used to solve the problem, which contains continuity equation and dynamic equation. In our model, the flood is characterized by its velocity, volume and duration.

Two questions proposed in the MCM problem A are tackled in the process of modeling. The first is how much flooding will occur in Rawls Creek and how far it will extend back. The second is whether the flood could be so massive downstream that water would reach up to the S.C. State Capitol Building.

To solve the partial differential equations, a proper difference schema and its iterative method are proposed. The corresponding Matlab program we developed to solve the equations simulates the flooding situations and some major conclusions are obtained. The answers to the two questions proposed are also worked out based on our assumptions about the parameter values and the conditions of the Saluda River.

The answer to the first question is: the distance which the flood can extend back in the Rawls Creek depends on the overflowing time, that is the time span during which the lake water overflows the dam through the breach. The longer the overflowing time is, the farther the flood will extend back. For example, if the overflowing time is assumed to be 30 minutes, then the flood will extend about 1000m in the Rawls Creek and the largest volume of the flood is about 40460 cubic meters per second.

The answer to the second question is: there exists a critical point of overflowing time. If the overflowing time is below that point, the Columbia City is safe; otherwise, it will get inundated. According to the computer simulation based on our assumptions, the critical point is 40 minutes and the flood will reach Columbia City after 83 minutes. The highest velocity is about 2.5 meters per second and the largest volume is about 10135 cubic meters per second. The flood can last 5 minutes with massive volumes.

By changing the parameter values in the assumptions, our model can be used to analyze the impact of the flooding in various situations. Based on the information thus obtained, the government can issue warnings of the potential flooding and take necessary measures in advance.

Restatement of the question

In this paper, the problem of flood caused by the breach of Saluda dam is addressed. The system of Saint Venant partial differential equations is used to solve the problem, which contains continuity equation and dynamic equation. In our model, the flood is characterized by its velocity, volume and duration.

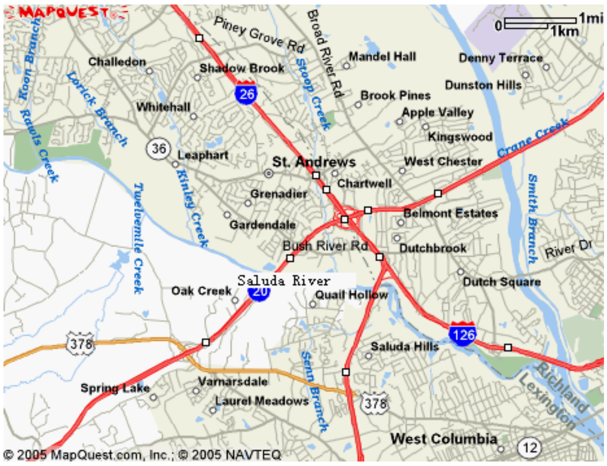
Two questions are tackled in the process of modeling. The first is how much flooding will occur in Rawls Creek and how far it will extend back. The second is whether the flood could be so massive downstream that water would reach up to the S.C. State Capitol Building.

Background Knowledge

To begin with, we must learn something about the water system mentioned in the given article.

LAKE MURRAY was constructed between 1927 and 1930. When completed, it was the largest power reservoir and Saluda Dam the largest earthen dam in the world. Today, Lake Murray stills supplies electric power, is the major source of drinking water for the city of Columbia, and has become the water playground for the Midlands.

Other data of Lake Murray and the dam available:
Dam Height 208 Feet /63.4 m at original Construction
Dam Length 1/2 Mile
Maximum Depth 190.7 Feet/58.1m
Mean Depth 41.5 Feet/12.6m
From the Internet, we also obtain the pictures below:



The picture shows clearly that Rawls Creek runs from north to south, while Saluda River from northwest to southeast. The former is a year-round stream that flows into the Saluda River a short distance downriver from the dam. Saluda River pours into the Congaree River at the spot nearby the city of Columbia.

The details of all the branches obtained by the rough measurement on the map:

Branch Name	Width(m)	the Distance between the Entrance of the Branch and the Dam(km)
Rawls Creek	8	4
Twelve Creek	8	8
Kinley Creek	8	10
Stoop Creek	8	15
Double Branch	8	25

All the information will be very helpful to our modeling.

Assumptions

1. Originally, the water system, including Saluda River and its branches, has a uniform velocity.
2. If the flood occurs, ignore the overflowing of the water, when it passes through the river and its branches.
3. The diffluent effect is grounded on the ratio between the branches' and the Saluda River's widths.
4. Saluda River doesn't have a large curvature, so we ignore the velocity loss when the flood passes through the curve bend.

Symbols and Definitions

1. x the distance between a specific spot on the river and the Saluda dam.

2. t time

3. T overflowing time, the time span during which the lake water overflows the dam through the breach

4. y water level

5. V the velocity of the flood, it is a vector.

6. A sectional area

7. Q the volume of the water flood flow

8. C Chézy coefficent

9. R hydraulic radius

10. K modulus of discharge $K = CA\sqrt{R}$

11. τ the step length of time

12. h the step length of the distance between a specific spot on the river and the Saluda dam (x)

13. a the ratio between the two step lengths, $a = \frac{\tau}{h}$.

Modeling

Step One: Ideal Model

This is to say, consider the Saluda River as a straight line, ignore other factors, such as branches, curve bend, the loss of flood's volume. Our ideal model is based on the system of Saint Venant equations which contains continuity equation and dynamic equation:

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = 0 \qquad (1)$$

$$-\frac{\partial y}{\partial x} = \frac{1}{g}\frac{\partial V}{\partial t} + \frac{V}{g}\frac{\partial V}{\partial x} + \frac{Q^2}{K^2} \qquad (2)$$

$\frac{1}{g}\frac{\partial V}{\partial t} + \frac{V}{g}\frac{\partial V}{\partial x}$ **is inertia term.**

Since the complete solution of the system is hard to get in hydraulics, in order to simplify the calculation, we first set

$$\frac{1}{g}\frac{\partial V}{\partial t} + \frac{V}{g}\frac{\partial V}{\partial x} = 0 \, , \qquad (3)$$

and ignore the effect of diffluence. So from the continuity equation and dynamic equation, we can get diffusion equation:

$$\frac{\partial Q}{\partial t} = V\frac{\partial Q}{\partial x} + D\frac{\partial^2 Q}{\partial x^2}. \quad (4)$$

difference schema:

$$V_{i,j+1} = \frac{\tau}{h} V_{i,j}\left(V_{i,j} - V_{i+1,j}\right) + V_{i,j} \quad (5)$$

$$\frac{Q_{i,j+1} - Q_{i,j}}{t} = V_{i,j}\frac{Q_{i+1,j} - Q_{i,j}}{h} + D\frac{Q_{i+1,j} - 2Q_{i,j} + Q_{i-1,j}}{h^2} \quad (6)$$

Because the total distance along Saluda River between the dam and the Columbia City is about 25 km, and the step length of the distance between a specific spot on the river and the Saluda dam (h) is relatively much larger than D, so we ignore the item of second partial derivative. Simplifying formula (5) and (6), we get the iterative format:

If $V_{i,j} > V_{i+1,j}$, $V_{i+1,j+1} = aV_{i+1,j}\left(V_{i,j} - V_{i+1,j}\right) + V_{i+1,j}$ **(7)**

If $V_{i,j} \le V_{i+1,j}$, $V_{i+1,j+1} = V_{i,j}$ **(8)**

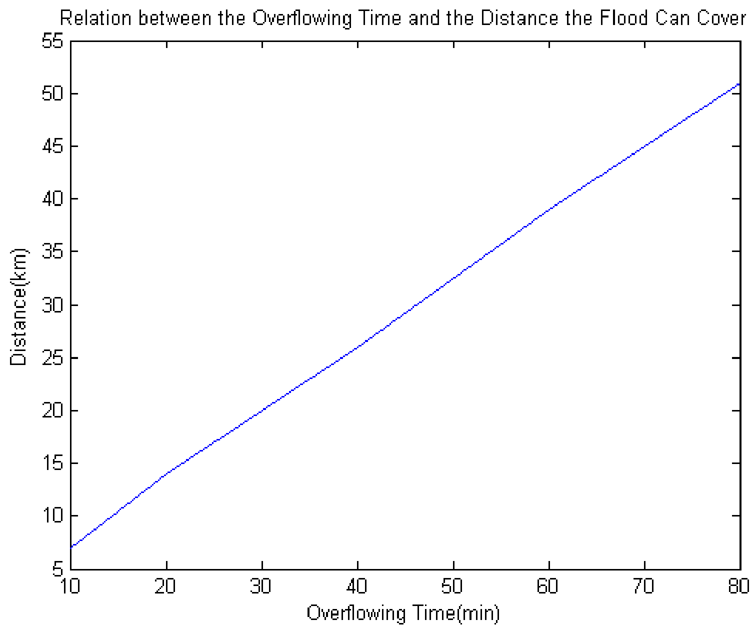
If $Q_{i,j} > Q_{i+1,j}$, $Q_{i+1,j+1} = aV_{i+1,j}\left(Q_{i,j} - Q_{i+1,j}\right) + Q_{i+1,j}$ **(9)**

If $Q_{i,j} \le Q_{i+1,j}$, $Q_{i+1,j+1} = Q_{i,j}$ **(10)**

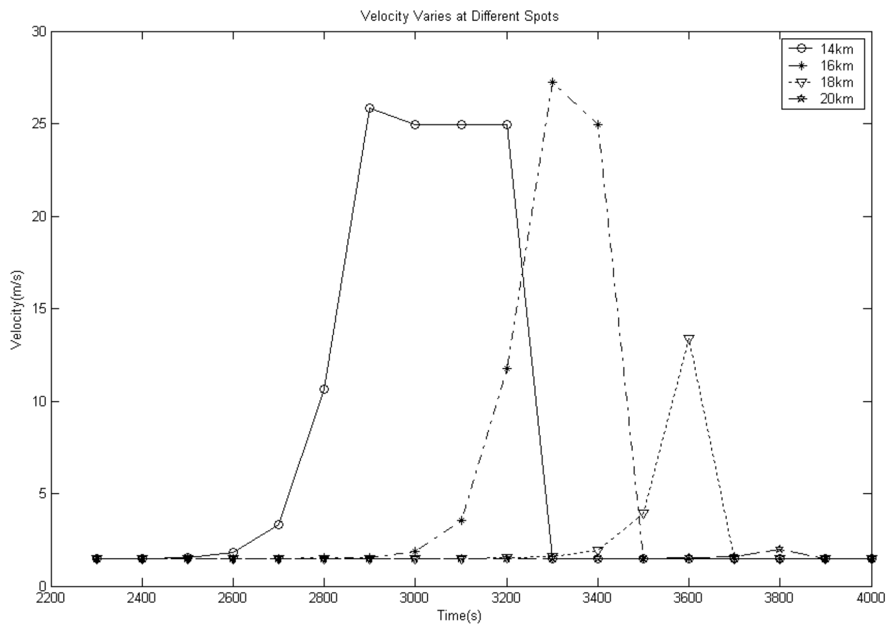
Initial condition: The time of dam demolishes we view as t=0, and except this dam spot (which we use x=0 to represent it), the velocity and the volume everywhere not changes, which equals initial value. The initial value we can get from the observation of the river. We measure the width of the Saluda River from the map and get an estimation value that is 80 meters. Then we premise the depth of the river is two meters. We can suppose the initial velocity of the water system to be 1.5 m/s, so the initial volume of the river can be got easily, which equals to 240 m^3/s, multiplying sectional area by velocity.

Boundary condition: Because of conservation of energy, we get boundary velocity equals $\sqrt{gH}$ at x=0,t=0 , H represents the height of the dam. It will last for some time (overflowing time). Then it will reduce to 1.5m/s. The flood sectional area is the height of the dam (63.4m) times the width (230m) at x=0,t=0. So the volume here equals the boundary velocity at x=0,t=0(24.96m/s) times the sectional area. In fact, it will last for some time (overflowing time), too. After that, it will reduce to the normal level.

Then we use MATLAB program, by adjusting parameters in the program, we get results that satisfy the realistic situation well. The two pictures below show our results:



The picture shows that the distance the flood can cover is in proportion to the overflowing time. Since the Columbia City is located about 25 miles downstream along Saluda River, the flood can reach it after about 40 minutes.



Information conveyed by the picture:

(1). To a specific point, the velocity rises dramatically to a high level when the flood arrives, after that, it will fall to the original level.

(2). The point which is near the dam will reach the highest level of the velocity earlier, and the high level can maintain longer.

(3). The highest levels of different points decrease along with the increase in x.

Conclusion:

There exists a critical point of overflowing time. If the overflowing time is below that point, the Columbia City is safe; otherwise, it will get inundated. According to the computer simulation based on our assumptions, the critical point is 40 minutes and the flood will reach Columbia City after 83 minutes. The highest velocity is about 2.5 meters per second. The largest volume is 16,322 m^3/s and it can last 10 minutes with massive volumes.

Step Two: Modified Model

We consider branches' influence on the volume. Actually, there are five branches along the river. They are Rawls Creek, Twelvemile Creek, Kinley Creek, Stoop Creek and Double Branch. The loss of volume will occur at the interjunction of rivers. In our model, the diffluence is in direct proportion to the width of two rivers. In this way, we modify the value that we get in the ideal model above. The modified value of the largest volume is 10,135 m^3/s and it can last 5 minutes with massive volumes.

If the overflowing time is long enough, for example, 40 minutes, flood will threaten Columbia. As we know, Rawls Creek is much nearer to the dam than the city. There is no need to discuss how far back it will extend in the Rawls Creek, because the flood will inundate the whole creek. So when we consider the first question, we will shorten the overflowing time. And see how long and how much the flood will extend in Rawls Creek.

The Saint Venant Equation is ineffective in solving the problem because the directions in which the two rivers (Saluda River and Rawls Creek) flow are different. Even though we assume the velocity is zero, we can't apply Saint Venant Equation to it. No river's velocity is zero! So we think out a solution: assume velocity is very small, such as 0.01 m/s and 0.001 m/s. Then we apply Saint Venant Equation. We want to find a specific spot in the Rawls Creek, the spot is nearest to the interjunction. Considering the actual velocity in Rawls Creek is 1.5 m/s in reverse direction, if velocity is about 1.5 m/s, it means that the velocity is zero and the flood will stop at the spot. Why we assume velocity is so small? Because it acts like zero to the value of flood's velocity, but it isn't zero, so we can apply Saint Venant Equation.

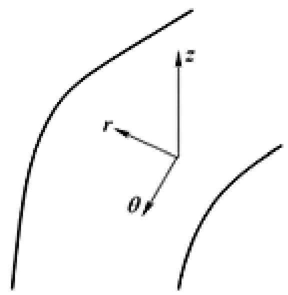
The following table show the result:

Assumed velocity / Overflowing time	0.001 m/s	0.0005 m/s	0.0001m/s
10 minutes	300 m	300 m	250 m
15 minutes	500 m	400 m	400 m
20 minutes	600 m	550 m	550 m
25 minutes	800m	650 m	650 m
30 minuftes	1000m	850 m	850 m

Conclusions:
(1). The distance which the flood can extend back in the Rawls Creek depends on the overflowing time of the flood, the longer the overflowing time is, the farther the flood will extend back.
(2). In a specific row, the data gradually become stable.

Generalization of the Model

A Method Dealing with the Bend Flow:



Cylindrical Coordinate System of the Bend Flow's Motion Equation

Taking steady flow as an example, we can obtain the motion equation of the circumfluence:

$$\begin{cases} u_r \frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} + u_z \frac{\partial u_r}{\partial z} - \frac{u_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \gamma\left(\Delta u_r - \frac{2}{r^2}\frac{\partial u_\theta}{\partial r} - \frac{u_r}{r^2}\right) \\ u_r \frac{\partial u_z}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta} + u_z \frac{\partial u_z}{\partial z} = -g - \frac{1}{\rho}\frac{\partial p}{\partial z} + \gamma \Delta u_z \\ \frac{\partial u_r}{\partial r} + \frac{\partial u_z}{\partial z} + \frac{u_r}{r} = 0 \end{cases} \tag{11}$$

Thereinto, $\Delta = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{\partial^2}{\partial z^2}$ **, v is the velocity of the bend flow in the main direction.**

For the curve bend whose bending radius is far bigger than the width of the section. The formula (11) can be simplified to be:

$$\begin{cases} \dfrac{u_\theta}{r}\dfrac{\partial u_r}{\partial \theta} = \dfrac{u_\theta^2}{r} - \dfrac{1}{\rho}\dfrac{\partial p}{\partial r} + \gamma\left(\dfrac{\partial^2 u_r}{\partial r^2} + \dfrac{\partial^2 u_r}{\partial z^2}\right) \\ \dfrac{u_\theta}{r}\dfrac{\partial u_z}{\partial \theta} = -g - \dfrac{1}{\rho}\dfrac{\partial p}{\partial z} + \gamma\left(\dfrac{\partial^2 u_z}{\partial r^2} + \dfrac{\partial^2 u_z}{\partial z^2}\right) \\ \dfrac{\partial u_r}{\partial r} + \dfrac{\partial u_z}{\partial z} = 0 \end{cases} \tag{12}$$

From the motion equations above, we can see that all the inertia terms are on the left side of the equations.

These two systems of partial differential equations have clear physical meaning and proved to be more practical at the cost of increase in computation.

Analysis of Sensitivity

We use explicit difference schema in the process of solving Saint Venant Equation. So the stability is very important. Adjusting parameters in program (set h=1 km and h=2 km), we find that the flood both covers about 25 km. Therefore, we think the algorithm stability is quite good.

Strengths and Weaknesses

Strengths:

(1). The ideal model is strong enough to enable us to analyze any situation of the flood, by changing the parameters in the Matlab programs.

(2). Our arithmetic is stable, it can tolerate a slight change of the initial value and the step length and has a good calculational effect.

Weaknesses:

Since our model has been developed on the basis of many assumptions, which ignore the effect of infiltration and overflowing, some characteristic of the flood can't be studied. For instance, the peak value of it can't be accurately calculated.

References

Renjun Zhao, 1984. Simulation of River Basin Hydrology. Beijing: Water

Resources and Electric Power Press.
Jianwei Hu, Huaimin Tang, 1999. Numerical Method of Differential Equation. Beijing: Science Press.
Dennis G. Zill, Michael R. Cullen, 2003. Differential Equations with Boundary-Value Problems. Beijing: China Machine Press.
Ellis, J., and Pender, G. Chute spillway design calculation. Proc. Inst . Civ. Engrs., 73(2), Thomas Telford Ltd., London, England, 299-312.
Roger, and Willi H. Hager. Supercritical bend flow. Journal of Hydra ulic Engineering ASCE, 123(3) : 208-218.