RUIN PROBABILITIES OF INSURANCE COMPANIES AND THE ORDINARY INTROGRO-DIFFERENTIAL EQUATION*

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Abstract When investigating the ruin probabilities of an insurance company, we find out the importance of understanding the second-order linear ordinary differential equation with non-local integral term. In the paper, we clearly expose the solution issue concerned with the equation. Firstly, we introduce the problem of the ruin probabilities of an insurance company and show the reason the ruin probabilities actually obey the second-order linear ordinary differential equation with non-local integral term. Second, we canvass the existence and uniqueness of the solution to the equation. By variable substitution and exchanging sequence of repeated integral, the non-local integral term can be simplified, and the equation can be transformed into a system of equations. In addition, the proof of existence and uniqueness of the solution of equations are completed here. At last, we analyze the structure of the solution, and provide the solution form of the equation.

Key words Ruin probability, Non-local integral term, Second-order differential equation

1 Introduction

During the recent global financial crisis in 2008, some insurance companies such as Yamato Life in Japan filed for bankruptcy. Even prior to the crisis, many insurance companies like Nissan Mutual Life, Toho Mutual Life, Tokyo Mutual Life and Kyoei Life Insurance Company declared bankruptcy one after another in Japan. The United States experienced "Black Monday" in 1989 after the collapse of a large number of insurance companies. Consequently how to reduce the ruin probabilities of insurance companies and protect the rights of insurance holders are a hot topic. Related research began with Gerber [1], and current research focuses on considering some factors within the risk model of the insurance's surplus or providing analytic solutions in special cases. Papers [2, 3] for example, take tax factor into account.

This paper introduces the classical risk model of the surplus for the insurance companies. Then it shows that if considering about investment, the ruin probability satisfies the second-order ordinary integro-differential equation. If not, the ruin probability satisfies the first-order ordinary integro-differential equation. The paper mainly studies the ordinary integro-differential equation thereafter.

2 Ruin Probabilities

In this section, we will introduce the re-

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alistic problem for the insurance companies and show how it is actually an ordinary integro-differential equation problem. Let $u > 0$ denote the initial capital, the premium income linearly increases proportionally, meaning the premium income during $[0, t]$ is ct. In addition, we use $S(t)$ to denote the amount of claim and W_t denotes the risky income at time t which the insurance company attains from the investment. Then the surplus of the insurance company at time t is R_t, which could be expressed as follows.

$$R_t = u + ct + \sigma W_t - S_t, \quad t \geq 0. \tag{2.1}$$

In (2.1), the claim $\{S(t)\}$ is a random variable dependent on time, called stochastic process. The stochastic process $\{S(t)\}$ consists of $\{N(t)\}$ (number of claims) and $\{Z_k, k = 1, 2, \cdots, N_t\}$ (the amount of each claim). If N is a Poisson process, then S is called a compound Poisson process. $\sigma > 0$ denotes the fluctuation ratio of underlying assets, and $\{W_t\}$ is a standard Brownian motion in (2.1).

Then we discuss the problem based on the following classical risk model.

$$R_t = u + ct + \sigma W_t - \sum_{k=1}^{N_t} Z_k, \quad t \geq 0. \tag{2.2}$$

$\{N_t\}$ is a Poisson process with parameter $\lambda > 0$; during period $(0, t)$ it counts the number of claims. $\{Z_k\}$, $k \geq 1$ is a non-negative sequence of i.i.d. random variables. Z_k denotes the amount of the kth claim. $\{N_t\}$, $\{Z_k\}$, $\{W_t\}$ are independent from each other. (2.2) is a continues time homogeneous strong Markov process.

From (2.2), we observe that two probabilities can lead to the bankruptcy of the insurance company. One is due to claims, the other to bad investment. As a result, Dufresne and Gerber [4] decompose the ruin probability in risk process (2.2) into two parts: the ruin probability due to the investment and the ruin probability due to the claims. Assuming that the ruin probability is second-order differentiable, they could obtain two different types of ruin probabilities expressed by series.

Let $a > 0$, define $\tau_a = \inf\{s : |W_s| = a\}$. When $x \in [-a, a]$, define two functions:

$$\begin{aligned} H(a,t,x) = \quad & (2\pi t)^{-1/2} \sum_{k=-\infty}^{+\infty} \Big[\exp\Big\{ -\frac{1}{2t}(x+4ka)^2 \Big\} - \\ & \exp\Big\{ -\frac{1}{2t}(x-2a+4ka)^2 \Big\} \Big], \end{aligned} \tag{2.3}$$

$$\begin{aligned} h(a,t) = \quad & \frac{1}{2\sqrt{2\pi}} a t^{-3/2} \sum_{k=-\infty}^{+\infty} \Big[(4k+1)\exp\Big\{ - \\ & \frac{a^2}{2t}(4k+1)^2 \Big\} + (4k-3)\exp\Big\{ -\frac{a^2}{2t}(4k-3)^2 \Big\} - \\ & (4k-1)\exp\Big\{ -\frac{a^2}{2t}(4k-1)^2 \Big\} \Big]. \end{aligned} \tag{2.4}$$

From [5], we have $P(W_s \in \mathrm{d}x, \tau_a > s) = H(a, s, x)\mathrm{d}x$, $P(\tau_a \in \mathrm{d}s) = h(a, s)\mathrm{d}s$.

Define that $T_u^0 = \inf\{t \geq 0; R_t < 0\}$, if $t \geq 0$, $R_t \geq 0$, then $T_u^0 = +\infty$. It is intuitive to observe that T_u^0 is the bankruptcy time. Let ruin probability $\psi(u)$ of (2.2) be defined as $\psi(u) = P(\inf_{t \geq 0} R_t < 0)$, then $\psi(u) = P(T_u < +\infty)$. Let $\mu = E[Z_1]$, $F(z) = P(Z_1 \leq z)$.

Since $\mu_s(t) = ES(t) = E\Big(E\big(S(t)|N(t)\big)\Big) = E\big(N(t)E(Z_1)\big) = EN(t)E(Z_1) = \lambda_t E(Z_1)$, when $\mu > 0$ and we assume that $E[R_t] = u + (c - \lambda\mu)t > 0$ (that is actually $c - \lambda\mu > 0$), then $\psi(u) < 1$. While when $c - \lambda\mu < 0$, $\lim_{t\to\infty} E[R_t] = -\infty$, then the bankruptcy will happen definitely, or $\psi(u) = 1$; and when $c - \lambda\mu = 0$, the bankruptcy must occur as well.

Here we use $\psi_d(u)$ to denote the ruin probability due to the improper investment and $\psi_s(u)$ to denote the ruin probability due to claims.

$$\psi(u) = \psi_d(u) + \psi_s(u). \tag{2.5}$$

Let $\{T_k\}, k \geq 1$, the jump time series of $\{N_t\}$, or the claim's time series. If $F(z)$ has a continuous density function at $[0, +\infty)$, then $P\big(\cup_{k=1}^{\infty} \{R_{T_k} = 0\}\big) = 0$, so

$$\begin{aligned} \psi_d(u) &= P(T_u^0 < +\infty, R_{T_u^0} = 0), \\ \psi_s(u) &= P(T_u^0 < +\infty, R_{T_u^0} < 0). \end{aligned} \tag{2.6}$$

Based on (2.6),

$$\psi_d(u) = \begin{cases} 0, & u < 0 \\ 1, & u = 0 \end{cases}$$
$$\psi_s(u) = \begin{cases} 1, & u < 0 \\ 0, & u = 0 \end{cases} \tag{2.7}$$

Theorem 2.1: *$u > 0$, and if $F(Z)$ has a continuous density function at $[0, +\infty]$, then the probability $\psi_d(u)$ satisfies the following integral equation.*

$$\begin{aligned} \psi_d(u) = & \frac{1}{2}\int_0^{+\infty} (\psi_d(ct) + \psi_d(2u+ct))\exp\{-\lambda t\}h(\frac{u}{\sigma}, t)\mathrm{d}t + \\ & \int_0^{+\infty} \lambda\exp\{-\lambda s\}\mathrm{d}s \int_{-u/\sigma}^{u/\sigma} H(\frac{u}{\sigma}, s, x)\mathrm{d}x \\ & \int_0^{u+cs+\sigma x} \psi_d(u + cs + \sigma x - z)\mathrm{d}F(z). \end{aligned} \tag{2.8}$$

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Proof: We use A_d to represent the bankruptcy due to the investment. $F_t = \sigma\{R_s, s \leq t\}$, $\sigma(R_s, s \leq t\}$ means all the possible information for the surplus R_s at time s. Let $M_t = E\big[I(A_d)|F_t\big]$, thus $\{M_t, t \geq 0\}$ is a F_t- martingale. Then let $T = \tau_{u/\sigma}\Lambda T_1$, Λ means to take the minimum of the two figures, we could have $P(T < +\infty) \leq P(T_1 < +\infty) = 1$. By optimal stopping theorem and properties of the strong Markov process-R_t, we obtain

$$\psi_d(u) = EM_0 = E[M_T] = E\Big[E\Big[I(A_d)|F_T\Big]\Big] = E\Big[\psi_d(R_T)\Big]. \tag{2.9}$$

Thus

$$\begin{aligned} \psi_d(u) \quad & = E[\psi_d(R_T)] \\ & = E[\psi_d(u + c\tau_{u/\sigma} + \sigma W_{\tau_{u/\sigma}})I(\tau_{u/\sigma} < T_1)] + \\ & E[\psi_d(u + cT_1 + \sigma W_{T_1} - Z_1)I(\tau_{u/\sigma} \geq T_1)] \\ & = I_1 + I_2. \end{aligned} \tag{2.10}$$

$$\begin{aligned} I_1 \quad & = E[\psi_d(u + c\tau_{u/\sigma} + \sigma W_{\tau_{u/\sigma}})I(\tau_{u/\sigma} < T_1)] \\ & = \tfrac{1}{2}E[\psi_d(2u + c\tau_{u/\sigma}) + \psi_d(c\tau_{u/\sigma})I(\tau_{u/\sigma} < T_1)]. \end{aligned} \tag{2.11}$$

Then from [6][1], we get

$$P(W_{\tau_a} = a, \tau_a \in \mathrm{d}t) = P(W_{\tau_a} = -a, \tau_a \in \mathrm{d}t) = \frac{1}{2}h(a, t)\mathrm{d}t.$$

From (2.11),

$$I_1 = \frac{1}{2}\int_0^{+\infty} \psi_d(ct) + \psi_d(2u + ct)\exp\{-\lambda t\}h(\frac{u}{\sigma}, t)\mathrm{d}t. \tag{2.12}$$

$$\begin{aligned} I_2 \quad & = E[\psi_d(u + cT_1 + \sigma W_{T_1} - Z_1)I(\tau_{u/\sigma} \geq T_1)] \\ & = \int_0^{+\infty} \lambda\exp\{-\lambda s\}\mathrm{d}s \int_0^{+\infty} \mathrm{d}F(z) \\ & \int_{-u/\sigma}^{u/\sigma} \psi_d(u + cs + \sigma x - z)H(\tfrac{u}{\sigma}, s, x)\mathrm{d}x \\ & = \int_0^{+\infty} \lambda\exp\{-\lambda s\}\mathrm{d}s \int_{-u/\sigma}^{u/\sigma} H(\tfrac{u}{\sigma}, s, x)\mathrm{d}x \\ & \int_0^{u+cs+\sigma x} \psi_d(u + cs + \sigma x - z)\mathrm{d}F(z).¶ \end{aligned} \tag{2.13}$$

The two following theorems' proofs are similar to that of Theorem 2.2 and 2.3 in [7].

Theorem 2.2: *If $F(z)$ has a continuous density function at $[0, +\infty]$, then $\psi_d(u)$ is second-order continuously differentiable at $(0, +\infty)$.*

Theorem 2.3: *If $F(z)$ has a continuous density function at $[0, +\infty]$, then $\psi_d(u)$ can satisfy the integro-differential equation below.*

$$\frac{1}{2}\sigma^2\psi_d''(u) + c\psi_d'(u) = \lambda\psi_d(u) - \lambda\int_0^u \psi_d(u - z)\mathrm{d}F(z), \tag{2.14}$$

where $u \in (0, \infty)$.

Similar to Theorem 2.2-2.3, we have the following two theorems.

Theorem 2.4: *If $F(z)$ has a continuous density function at $[0, +\infty]$, then $\psi_s(u)$ is second-order continuously differentiable at $(0, +\infty)$.*

Theorem 2.5: *If $F(z)$ has a continuous density function at $[0, +\infty]$, then $\psi_s(u)$ can satisfy the integro-differential equation below.*

$$\frac{1}{2}\sigma^2\psi_s''(u) + c\psi_s'(u) = \lambda\psi_s(u) - \lambda\int_0^u \psi_s(u - z)\mathrm{d}F(z) - \lambda(1 - F(u)), \tag{2.15}$$

where $u \in (0, \infty)$.

Proposition 2.1: *If $F(z)$ has a continuous density function at $[0, +\infty]$, then $\psi_d(u)$ is also continuous at $[0, +\infty]$.*

Proof: From Theorem 2.1, we know that $\psi_d(0^+) = \psi_d(0)$. And because of $P(T_{0^+}^0 = T_0^0 = 0) = 1$ [7][2], while R_t is continuous at the right, thus the following

[1] proposition 2.8.3 of Port and Stone(1978)
[2] Proposition 4.1 of Wang and Wu(2000)

equality could be provided by the Uniform Convergence Theorem,

$$\begin{aligned}\lim_{u\to 0}\psi_d(u) &= \lim_{u\to 0} E[I(T_u^0 < +\infty, R_{T_u^0} = 0)] \\ &= E[\lim_{u\to 0} I(T_u^0 < +\infty, R_{T_u^0} = 0)] \\ &= E[I(T_0^0 < +\infty, R_0 = 0)] = 1 = \psi_d(0).\P\end{aligned}$$

Similarly we propose Proposition 2.2,

Proposition 2.2: *If $F(z)$ has a continuous density function at $[0, +\infty]$, then $\psi_s(u)$ is also continuous at $[0, +\infty]$.*

Then we add both sides of (2.14) with (2.15),

$$\begin{aligned}&\tfrac{1}{2}\sigma^2(\psi_d(u) + \psi_s(u))'' + c(\psi_d(u) + \psi_s(u))' \\ &= \lambda(\psi_d(u) + \psi_s(u)) \\ &-\lambda \textstyle\int_0^u (\psi_d(u-z) + \psi_s(u-z))\mathrm{d}F(z) - \lambda(1 - F(u)),\end{aligned} \tag{2.16}$$

then

$$\frac{1}{2}\sigma^2\psi''(u) + c\psi'(u) = \lambda\psi(u) - \lambda\int_0^u \psi(u-z)\mathrm{d}F(z) - \lambda(1-F(u)). \tag{2.17}$$

If $F(z)$ is differentiable, $F'(z) = f(z)$,

$$\frac{1}{2}\sigma^2\psi''(u) + c\psi'(u) - \lambda\psi(u) + \lambda\int_0^u \psi(u-z)f(z)\mathrm{d}z = \lambda(F(u)-1). \tag{2.18}$$

Since σ is a positive constant,

$$\psi''(u) + \frac{2c}{\sigma^2}\psi'(u) - \frac{2\lambda}{\sigma^2}\psi(u) + \frac{2\lambda}{\sigma^2}\int_0^u \psi(u-z)f(z)\mathrm{d}z = \frac{2\lambda}{\sigma^2}(F(u)-1). \tag{2.19}$$

(2.19) is the second-order linear ordinary differential equation with non-local integral term we will discuss in the remaining chapters.

If the company do not consider about investment, i.e. $\sigma W_t = 0$, $R_t = u + ct - \sum_{k=1}^{N_t} Z_k$, the classical risk model could be simplified. Thus, the ruin probability $\psi(u)$ satisfies: [8][3]

$$\psi'(u) - \frac{\lambda}{c}\psi(u) + \frac{\lambda}{c}\int_0^u \psi(u-z)f(z)\mathrm{d}z = \frac{\lambda}{c}(F(u)-1) \quad u \in (0, \infty). \tag{2.20}$$

The order is one lower. However, it is not easy to get their analytic solutions for either first-order or second-order equation. The analytic solutions could only be obtained in some specific cases.

[3] p165-166 of Gerber and Shiu(1997)

Both equations (2.19,2.20) don't consider the interest rate, if we consider the interest rate $I_t = rt$ [7], then the ruin probability $\psi(u)$ satisfies (3.5) in the following chapter,

$$\psi'(u) - \frac{r+\lambda}{c}\psi(u) + \frac{\lambda}{c}\int_0^u \psi(u-z)f(z)\mathrm{d}z = \frac{\lambda}{c}(F(u)-1). \tag{2.21}$$

We mentioned before that we could solve the ruin probability $\psi(u)$ analytically only in some special cases. The paper skips this part, but interested readers could read [7, 9, 10] for reference.

3 The Existence and Uniqueness of the Solution

In this section, we discuss the existence and uniqueness of the solution to the second-order linear ordinary differential equation with a non-local integral term that the ruin probabilities satisfied. Initially, we propose the general form of the equation:

$$\begin{cases} y''(x) + p(x)y'(x) + q(x)y(x) + \lambda(x)\int_0^x y(x-t)m(t)\mathrm{d}t = f(x) \\ y(0) = y_0, y'(0) = y_1, \end{cases} \tag{3.1}$$

where $p(x)$, $q(x)$, $\lambda(x)$ are non-constant coefficients. Then we consider the existence and uniqueness of the solution to problem(3.1). Let $y'(x) = z(x)$, we get

$$\begin{cases} y'(x) = z(x) \\ z'(x) = -p(x)z(x) - q(x)y(x) - \lambda(x)\int_0^x y(x-t)m(t)\mathrm{d}t + f(x). \end{cases} \tag{3.2}$$

Integrating both sides of (3.2), we obtain

$$z(x) = -\int_0^x [p(u)z(u) + q(u)y(u) - f(u)]\mathrm{d}u$$
$$-\int_0^x \lambda(u)[\int_0^u y(u-t)m(t)\mathrm{d}t]\mathrm{d}u + c.$$

c is determined by $z(0)$, $c = z(0) = y_1$, and thus we have

$$\int_0^u y(u-t)m(t)\mathrm{d}t = \int_0^u y(w)m(u-w)\mathrm{d}w = \int_0^u y(t)m(u-t)\mathrm{d}t, \tag{3.3}$$

$$\int_0^x \lambda(u)\Big[\int_0^u y(u-t)m(t)\mathrm{d}t\Big]\mathrm{d}u = \int_0^x \lambda(u)\Big[\int_0^u y(t)m(u-t)\mathrm{d}t\Big]\mathrm{d}u. \tag{3.4}$$

By exchanging the sequence of repeated integral, (3.4) becomes

$$\int_0^x y(t)\mathrm{d}t \int_t^x \lambda(u)m(u-t)\mathrm{d}u = \int_0^x y(u)\mathrm{d}u \int_u^x \lambda(t)m(t-u)\mathrm{d}t.$$

Let $\int_u^x \lambda(t)m(t-u)\mathrm{d}t = M(x,u)$. We will discuss at $0 \le x \le b$, but the conclusion could extend to $a \le x \le b$, $a \le 0$. Let

$$\boldsymbol{Y}(x) = \begin{pmatrix} y(x) \\ z(x) \end{pmatrix}, \quad \boldsymbol{\eta} = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}, \quad \boldsymbol{Y}(0) = \begin{pmatrix} y(0) \\ z(0) \end{pmatrix} = \boldsymbol{\eta}.$$

Then the equations' system (3.2) will be transformed to

$$\begin{aligned} \boldsymbol{Y}'(x) = & \begin{pmatrix} 0 & 1 \\ -q(x) & -p(x) \end{pmatrix} \boldsymbol{Y}(x) \\ & - \lambda(x) \textstyle\int_0^x \begin{pmatrix} 0 & 0 \\ m(x-t) & 0 \end{pmatrix} \boldsymbol{Y}(t)\mathrm{d}t + \begin{pmatrix} 0 \\ f(x) \end{pmatrix}. \end{aligned} \tag{3.5}$$

By integrating both sides, we get

$$\begin{aligned} \boldsymbol{Y}(x) = & \quad \boldsymbol{\eta} + \textstyle\int_0^x \left[\begin{pmatrix} 0 & 1 \\ -q(x) & -p(x) \end{pmatrix} \boldsymbol{Y}(x) + \begin{pmatrix} 0 \\ f(u) \end{pmatrix} \right] \mathrm{d}u \\ & - \textstyle\int_0^x \begin{pmatrix} 0 & 0 \\ M(x,u) & 0 \end{pmatrix} \boldsymbol{Y}(u)\mathrm{d}u. \end{aligned}$$

Assuming

$$\boldsymbol{A}(x,u) = \begin{pmatrix} 0 & 1 \\ -q(u) - M(x,u) & -p(u) \end{pmatrix}, \boldsymbol{F}(u) = \begin{pmatrix} 0 \\ f(u) \end{pmatrix},$$

are continuous at $0 \le u \le x$, $0 \le x \le b$, we have

$$\boldsymbol{Y}(x) = \boldsymbol{\eta} + \int_0^x [\boldsymbol{A}(x,u)\boldsymbol{Y}(u) + \boldsymbol{F}(u)]\mathrm{d}u. \tag{3.6}$$

Similar to the proof of [11][4], the existence and uniqueness of the solution to the equations could be deduced.

If $\begin{pmatrix} 0 & 1 \\ -q(x) & -p(x) \end{pmatrix}$ is a 2×2 matrix, $\begin{pmatrix} 0 \\ f(x) \end{pmatrix}$ is a two dimension vector, $\lambda(x)$ is a function of x, $\begin{pmatrix} 0 & 0 \\ m(x-t) & 0 \end{pmatrix}$ is a 2×2 matrix. They are all continuous at $0 \le x \le b$, $0 \le t \le x$. Then for 0 at $a \le x \le b$ and one arbitrary constant vector $\boldsymbol{\eta} = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$, the system of equations

$$\begin{aligned} \boldsymbol{Y}'(x) = & \begin{pmatrix} 0 & 1 \\ -q(x) & -p(x) \end{pmatrix} \boldsymbol{Y}(x) \\ & - \lambda(x) \textstyle\int_0^x \begin{pmatrix} 0 & 0 \\ m(x-t) & 0 \end{pmatrix} \boldsymbol{Y}(t)\mathrm{d}t + \begin{pmatrix} 0 \\ f(x) \end{pmatrix} \end{aligned}$$

has one unique solution $\boldsymbol{\phi}(x)$, which is defined in the whole interval $0 \le x \le b$ and satisfies the initial condition, $\boldsymbol{\phi}(0) = \boldsymbol{\eta}$.

[4] Five propositions from P68-P72

4 The Structure of the Solution Space

In the last section, we will discuss the structure of the solution space to the above equations from (3.5), and let

$$\boldsymbol{A}(x) = \begin{pmatrix} 0 & 1 \\ -q(x) & -p(x) \end{pmatrix},$$

$$\boldsymbol{B}(x,t) = \begin{pmatrix} 0 & 0 \\ m(x-t) & 0 \end{pmatrix},$$

$$\boldsymbol{F}(x) = \begin{pmatrix} 0 \\ f(x) \end{pmatrix}.$$

Then

$$\boldsymbol{Y}'(x) = \boldsymbol{A}(x)\boldsymbol{Y} - \lambda(x) \int_0^x \boldsymbol{B}(x,t)\boldsymbol{Y}(t)\mathrm{d}t + \boldsymbol{F}(x). \tag{4.1}$$

The related homogeneous equation is

$$\boldsymbol{Y}'(x) = \boldsymbol{A}(x)\boldsymbol{Y} - \lambda(x) \int_0^x \boldsymbol{B}(x,t)\boldsymbol{Y}(t)\mathrm{d}t. \tag{4.2}$$

The linear combination of solutions to (4.2) still satisfy the equation.

Several theorems are provided without proof:

Theorem 4.1: *If vector functions* $\mathbf{Y}_1(x)$,$\mathbf{Y}_2(x)$ *are linearly correlated in* $a \le x \le b$, *then their Wronskian* $\boldsymbol{W}(x) = 0$, $a \le x \le b$.

Theorem 4.2: *If the solutions to problem (4.2),* $\mathbf{Y}_1(x)$,$\mathbf{Y}_2(x)$, *are linearly independent, then their Wronskian* $\boldsymbol{W}(x) \ne 0$, $a \le x \le b$.

Theorem 4.3: *(4.2) must have two independent solutions* $\mathbf{Y}_1(x)$, $\mathbf{Y}_2(x)$.

Theorem 4.4: *If* $\mathbf{Y}_1(x)$, $\mathbf{Y}_2(x)$ *are two independent solutions to (4.2), then any solution* $\boldsymbol{Y}(x)$ *to problem(4.2) could be expressed as* $\boldsymbol{Y}(x) = c_1\mathbf{Y}_1(x) + c_2\mathbf{Y}_2(x)$, *here* c_1,c_2 *are determined constants.*

Thus, we could infer that the maximum number of solutions to the problem (4.2) is 2, and all solutions to the problem construct a two dimensional linear space.

Theorem 4.3-4.4 could be expressed as

Theorem 4.5: *There exists a fundamental solution matrix* $\boldsymbol{\phi}(x)$ *for (4.2), if* $\boldsymbol{\psi}(x)$ *is one arbitrary solution to (4.2), then* $\boldsymbol{\psi}(x) = \boldsymbol{\phi}(x)\boldsymbol{C}$*.* $\boldsymbol{C}$ *is determined to be a constant vector.*

Two simple properties for (4.2):

Property 4.1: *If* $\boldsymbol{\psi}(x)$ *is the solution to (4.1), and* $\boldsymbol{\phi}(x)$ *is the solution to its related homogenous equations (4.2), then* $\boldsymbol{\psi}(x) + \boldsymbol{\phi}(x)$ *is the solution to (4.1).*

Property 4.2: *If* $\tilde{\boldsymbol{\psi}}(x)$ *and* $\tilde{\boldsymbol{\phi}}(x)$ *are two solutions to (4.1), then* $\tilde{\boldsymbol{\psi}}(x) - \tilde{\boldsymbol{\phi}}(x)$ *is the solution to (4.2).*

From Property 4.2 and Theorem 4.5, we know if $\boldsymbol{\phi}(x)$ is a fundamental solution matrix to (4.2). If $\bar{\boldsymbol{\phi}}(x)$ is a specific solution to (4.1), then any solution $\boldsymbol{\psi}(x)$ to the problem (4.1 could be expressed as $\boldsymbol{\psi}(x) = \boldsymbol{\phi}(x)\boldsymbol{C} + \bar{\boldsymbol{\phi}}(x)$. $\boldsymbol{C}$ is a determined constant vector here.

In the end, we briefly describe our contributions to the literature. For the ruin probabilities problem of insurance companies, we provide detailed explanation of its related ordinary differential equation with non-local integral, i.e, how the solutions should be for this specific equation.

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