EXISTENCE OF A SHOCK WAVE IN A ONE-DIMENSIONAL RADIATION HYDRODYNAMIC SYSTEM*

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Abstract In this paper, we mainly study shock waves in a one-dimensional radiation hydrodynamic system. By using the Rankine-Hugoniot condition and entropy condition, this problem can be formulated as an initial boundary problem with a free boundary for radiation hydrodynamic system. First, we transform this free boundary to the fixed one by using change of variables involving unknowns. Then we investigate the existence and uniqueness of the solution to the initial boundary problem for this nonlinear system. For this problem, we first construct an approximate solution by using the compatibility conditions of the data. Then we use the Picard iteration and the Newton iteration for this nonlinear system respectively to construct a sequence of approximate solutions. By using a series of estimates and a compactness argument, we get the convergence of the sequence of approximate solutions. The limit of this sequence gives a shock wave of the original radiation hydrodynamic system.

Key words one-dimensional radiation hydrodynamic systems, shock waves, existence

2000 MR Subject Classification 78A40, 35L67, 74G20

1 Introduction

At high temperature, radiation has significant influence on the motion of fluids. The applications of the radiation hydrodynamic systems are very broad, including such diverse astrophysical phenomena as waves and oscillations in stellar atmospheres and envelops, nonlinear stellar pulsation, supernova explosions, stellar winds and many others. It has also direct application in other areas, for instance to the physics of laser fusion and reentry of vehicles. It's meaningful for us to investigate the mechanism of radiation phenomena and the influence of radiation to the character exchange of data of fluids as energy, momentum, temperature and so on. Radiation hydrodynamics finds a wide range of application in scientific research, civil economy and military construction. The research on physical modeling and numerical simulation of radiation hydrodynamics developed in the world for many years, and some important results have already got ([14, 15]). Then, the research on

*The work is partially supported by a joint project from the NSAF of China under the grant 10676020, and a key project of NSFC under the grant 10531020.

the mathematical theory of radiation hydrodynamics possess significant reference effect for us to understand the nature of models and well designed numerical scheme. In radiation hydrodynamics, when we neglect the affection of viscidity and heat exchange, the density, velocity and energy of fluids can be described by system of the Euler equations (hyperbolic) coupled with a transport equation (Boltzmann equation). Therefore, the study of mathematical theory of radiation hydrodynamics is of great importance both from the mathematical theory and application point of view.

We use $I(x,t,\nu,\Omega)$ to denote the spcicific intensity of radiation (at time t) at space point x, with frequency $\nu > 0$ in a direction Ω, then the system of partial differential equations of one-dimensional isentropic radiation hydrodynamics (cf. [14, 15]), can be represented as following:

$$\begin{cases} \frac{1}{c}\frac{\partial I(\nu,\Omega)}{\partial t} + \Omega_1 \frac{\partial I(\nu,\Omega)}{\partial x} = S(\nu) - \sigma_a(\nu) I(\nu,\Omega) \\ \qquad + \int_0^\infty d\nu' \int_{S^2} \left(\frac{\nu}{\nu'} \sigma_s(\nu' \to \nu, \Omega' \cdot \Omega) I(\nu',\Omega') - \sigma_s(\nu \to \nu', \Omega \cdot \Omega') I(\nu,\Omega) \right) d\Omega' \\ \rho_t + (\rho u)_x = 0 \\ (\rho u + \frac{1}{c^2} F_r)_t + (\rho u^2 + P + P_r)_x = 0 \end{cases} \tag{1.1}$$

where c and $S(\nu)$ represent light speed and the rate of energy emission due to spontaneous processes, Ω_1 denotes the projection of Ω along the x-axis. We can make it equals to 1 without losing generality. $\sigma_a(\nu) = \sigma_a(x,t,\nu,\rho)$ denotes the absorption coefficient. Similar to absorption, a photon can undergo scattering interactions with matter, and the scattering interaction serves to change the photon's characteristics ν' and Ω' to a new set of characteristics ν and Ω, this leads to the definition of the "differential scattering coefficient" $\sigma_s(\nu' \to \nu, \Omega' \cdot \Omega)$. $\rho(t,x)$ is the density, u is the speed while $P = P(\rho)$ is the pressure. F_r and P_r represent the radiative flux and the radiative pressure tensor respectively can be defined by

$$\begin{cases} F_r = \int_0^\infty d\nu \int_{S^2} \Omega_1 I(\nu,\Omega) d\Omega, \\ P_r = \frac{1}{c} \int_0^\infty d\nu \int_{S^2} \Omega_1^2 I(\nu,\Omega) d\Omega. \end{cases} \tag{1.2}$$

There are abundant results about discontinuous solutions theory for one-dimensional quasilinear hyperbolic conservation laws (cf. [2, 10, 11, 12, 13, 17]). But there are few mathematical results on the radiation hydrodynamics system (1.1)-(1.2) due to its complexity. Recently, Jiang-Zhong in [19] obtained the local existence of C^1 solutions for the Cauchy problems of the general radiation hydrodynamical system, and showed the finite-time blowup of C^1 solutions under the assumption that the initial data is large. Rohde and Yong [16] constructed the existence of entropy solutions and its nonrelativistic limit towards one kind of simplified nonlinear balance equation coupled with radiation transport equation. Given some simplified assumptions towards radiation hydrodynamics, Kawashima [6, 7] used nonlinear hyperbolic equations coupled with elliptical equation to describe the motion of one-dimensional radiation fluids with some other people. Ito [4] showed us the existence of bounded variation solutions to it then, and later on, Kawashima with some others built stability of some nonlinear waves in [8, 9]. In [1], Anile, Blokhin and Trakhinin investigated a mathematical model for radiation hydrodynamics. Then, they got the existence of of local smooth solutions to the Cauchy problem for the equations through symmetrization. But the research on the theory of discontinuous solutions to the radiation hydrodynamic systems which have vital application background is still in lack, especially for the transmission and disturbance of singularity of waves. To obtain the existence of a shock wave, we need to overcome the complex structure of the system, especially from the radiation terms in the Euler equations (1.1), which requires some new ideas and new ingredients in the proof.

The rest of this paper is organized as follows: In §2, first, By using the Rankine-Hugoniot condition and Lax entropy condition, we know this problem can be formulated as an initial boundary problem with a free boundary for radiation hydrodynamic system. Then, we transform this free boundary to the fixed one by using change of variables involving unknowns. Later on we investigate the existence and uniqueness of the solution to the initial boundary problem for this nonlinear system. Therefore we first construct an approximate solution in §3. Then we use the Picard iteration and the Newton iteration for this nonlinear system respectively to construct a sequence of approximate solutions. By using a series of estimates and a compactness argument, we get the convergence of the sequence of approximate solutions with the local existence of shock waves of this radiation hydrodynamic system.

2 Problem introduction & main results

2.1 Problem introduction

Using (1.2), we can rewrite (1.1) as

$$U_t + A(U) \cdot U_x = B(U). \tag{2.1}$$

Where $m = \rho u$, and

$$U = \begin{pmatrix} I \\ \rho \\ m \end{pmatrix}, \quad A(U) = \begin{pmatrix} c & 0 & 0 \\ 0 & 0 & 1 \\ 0 & a^2 - \frac{m^2}{\rho^2} & \frac{2m}{\rho} \end{pmatrix}, \quad B(U) = \begin{pmatrix} f \\ 0 \\ g \end{pmatrix}, \tag{2.2}$$

with $a = \sqrt{P'(\rho)}$ is the velocity of sound.

Set

$$F(U) = \left(cI, m, \frac{m^2}{\rho} + P \right)^T \tag{2.3}$$

then, (1.1) can be also expressed as the balance equations' form below,

$$U_t + F(U)_x = B(U). \tag{2.4}$$

We will consider about the shock wave solutions to the Riemann problem for equations (2.4) with the following piecewise smooth initial condition,

$$U(0,x) = \begin{cases} U_{+,0}(x), & x > 0 \\ U_{-,0}(x), & x < 0 \end{cases}. \tag{2.5}$$

On the assumption that the equations with odd degree corresponding to (2.4) with the constant initial condition,

$$U_0(0,x) = \begin{cases} U_{+,0}(0), & x > 0 \\ U_{-,0}(0), & x < 0 \end{cases}$$

when the solutions to it are shock waves, we will discuss the discontinuous solutions, mainly on the local existence of shock waves of problem (2.4)-(2.5).

The eigenvalues of $A(U)$ can be got by simple calculation:

$$\lambda_1 = u - a, \quad \lambda_2 = u + a, \quad \lambda_3 = c.$$

For c represents the light speed, so we can assume that $\lambda_1 < \lambda_2 < \lambda_3$. Let l_k and r_k be the corresponding left and right eigenvectors of $A(U)$ with respect to λ_k for each $k \in \{1, 2, 3\}$,

$$l_1 = (0, -(u+a), 1), \quad l_2 = (0, a-u, 1), \quad l_3 = (1, 0, 0)$$

$$r_1 = (0, -\frac{1}{2a}, \frac{a-u}{2a})^T, \quad r_2 = (0, -\frac{1}{2a}, -\frac{u+a}{2a})^T, \quad r_3 = (1, 0, 0)^T$$

with normalization

$$l_i \cdot r_j = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}.$$

Suppose that (A1) and (A2) are satisfied here:

(A1) The Riemann problem

$$\begin{cases} \partial_t U_0 + \partial_x F(U_0) = 0 \\ U_0(0,x) = \begin{cases} U_{+,0}(0_+), & x > 0 \\ U_{-,0}(0_-), & x < 0 \end{cases} \end{cases} \tag{2.6}$$

has the shock wave solution:

$$U_0(t,x) = \begin{cases} U_0^+, & x > \sigma t \\ U_0^-, & x < \sigma t \end{cases}. \tag{2.7}$$

$U_0^{\pm}$ satisfies the Rankine-Hugoniot condition on $\{x = \sigma t\}$

$$(U_0^+ - U_0^-)\sigma = F(U_0^+) - F(U_0^-).$$

If $(\sigma, U_0^+(0,0), U_0^-(0,0))$ is the 1-shock wave, then it also satisfies the Lax entropy condition:

$$\begin{cases} \sigma < \lambda_1(U_0^-) < \lambda_2(U_0^-) < \lambda_3 \\ \lambda_1(U_0^+) < \sigma < \lambda_2(U_0^+) < \lambda_3 \end{cases}$$

(A2) The shock wave mentioned before satisfies the stability condition:

$$[U_0(0)] = U_{+,0}(0) - U_{-,0}(0), r_2(U_{+,0}(0)), r_3(U_{+,0}(0))$$

are linear independence.

Remark 2.1: Just as Lax pointed out in [10], the stability condition is satisfied when shock wave is weak.

Cauchy problem (2.4)-(2.5) can be written as the following matrix form

$$\begin{cases} U_t + A(U)\cdot U_x = B(U) \\ U(0,x) = \begin{cases} U_{+,0}(x), & x>0 \\ U_{-,0}(x), & x<0 \end{cases}. \end{cases} \tag{2.8}$$

Because we will discuss about the local solutions to the problem (2.8), we could suppose that when $|x| \geq R$, $U_{\pm,0}(x)$ is constant for the limited transmitting speed of hyperbolic equation. Given a small neighborhood $\omega \subset \{t=0\}$ of the origin, suppose $\aleph$ is a determinacy domain of ω for the Cauchy problem (2.8).

We will construct piecewise C^1 weak solution to (2.9)

$$\left\| \begin{array}{l} U(t,x) = \begin{cases} U_+(t,x), & x > x(t) \\ U_-(t,x), & x < x(t) \end{cases} \\ x(0) = 0 \end{array} \right. \tag{2.9}$$

the weak solution U satisfies equations (2.8) in $\aleph^{\pm} = \{\pm(x - x(t)) > 0\}$ and the Rankine-Hugoniot condition $(U_+ - U_-)x'(t) = F(U_+) - F(U_-)$ on $\{x = x(t)\}$, also it satisfies 1-shock wave entropy condition:

$$\begin{cases} x'(t) < \lambda_1(U_-(t,x(t))) < \lambda_2(U_-(t,x(t))) < \lambda_3 \\ \lambda_1(U_+(t,x(t))) < x'(t) < \lambda_2(U_+(t,x(t))) < \lambda_3 \end{cases}$$

2.2 Problem transform

As $(U_+, U_-, x(t))$ of problem (2.8) is unknown, so we want to discuss the existence of solution to the initial boundary problem with a free boundary for radiation hydrodynamic system.

For $x'(t) < \lambda_1^- < \lambda_2^- < \lambda_3$, then $\aleph^- = \{x < x(t)\}$ is the determinacy domain of $\{x < 0\}$ for problem (2.8). We can get the value of U_- in $\aleph$ first, therefore, we extend $U_{-,0}(x)$ to $\{x \geq 0\}$ as C^1 continuous, and we denote it by $\tilde{U}_{-,0}(x)$.

To solve Cauchy problem

$$\begin{cases} \tilde{U}_t + A(\tilde{U})\cdot \tilde{U}_x = B(\tilde{U}) \\ \tilde{U}(0,x) = \tilde{U}_{-,0}(x) \end{cases} \tag{2.10}$$

we get the local solution $\tilde{U}_-(t,x) \in C^1(0 \leq t < T, x \in I\!R)$. Let $U_-|_{\aleph^-} = \tilde{U}_-|_{\aleph^-}$, the value of U_- is independent of the extension in $\aleph^-$ for $\aleph^-$ is the determinacy domain of $\{x < 0\}$.

Thus, we can assume that U_- is known to problem (2.8)-(2.9), then we discuss the following problem

$$\begin{cases} \partial_t U_+ + A(U_+)\cdot \partial_x U_+ = B(U_+) \qquad \text{in} \quad \aleph^+ = \{x > x(t)\} \\ (U_+ - U_-)\cdot x'(t) - (F(U_+) - F(U_-)) = 0 \qquad \text{on} \quad \{x = x(t)\} \\ U_+|_{t=0} = U_{+,0}(x) \qquad x > 0 \end{cases} \tag{2.11}$$

it's a free boundary value problem for U_+ and $x(t)$ are unknown in (2.11).

In order to transform this problem into the fixed boundary case, we perform the transformation

$$\begin{cases} \tilde{x} = x - x(t) \\ \tilde{t} = t. \end{cases}$$

Letting $\tilde{\aleph}^+ = \{x > 0\}$. We still use symbols (t, x) and $\aleph^+$ to represent $(\tilde{t}, \tilde{x})$ and $\tilde{\aleph}^+$, then obtain the initial boundary problem in fixed area,

$$\begin{cases} \partial_t U_+ + (A(U_+) - x'(t)I) \cdot \partial_x U_+ = B(U_+) & (t,x) \in (t > 0, x > 0) \\ (U_+ - U_-) \cdot x'(t) - (F(U_+) - F(U_-)) = 0 & \text{on} \quad \{x = 0\} \\ x(0) = 0 \\ U_+|_{t=0} = U_{+,0}(x), \qquad x > 0 \end{cases} \tag{2.12}$$

2.3 Compatibility conditions

In order to discuss (2.12), let's study compatibility conditions.

(1) the zero-th order compatibility condition:

Since the boundary condition in (2.12) must be valid at $\{x = t = 0\}$, the zero-th order compatibility condition for the problem (2.12) is

$$x'(t)|_{t=0}[U_0(0)] = [F(U_0)] \tag{2.13}$$

where $[U_0(0)] = U_{+,0}(0) - U_{-,0}(0)$.

It's the supposition mentioned before $\sigma(U_{+,0}(0) - U_{-,0}(0)) = F(U_{+,0}(0)) - F(U_{-,0}(0))$, it can be got from Rankine-Hugoniot condition from (A1).

(2) the first order compatibility condition:

Differentiating the boundary condition in (2.12) with respect to t, and evaluating the result at $\{x = t = 0\}$, we obtain

$$x''(0)[U_0(0)] + \sigma[\partial_t U|_{t=0}] - [A(U)\partial_t U|_{t=0}] = 0. \tag{2.14}$$

On the other hand, from the equation and initial data in (2.12) we have

$$\partial_t U_+(0,0) = (\sigma I - A(U_{+,0}(0)))d_x U_{+,0}(0) + B(U_{+,0}(0)). \tag{2.15}$$

Substituting (2.15) into (2.14), it follows the first order compatibility condition for the problem (2.12):

$$\begin{aligned} x''(0)[U_0(0)] + (\sigma I - A(U_{+,0}(0)))^2 d_x U_{+,0}(0) = & -(\sigma I - A(U_{+,0}(0))) \cdot B(U_{+,0}(0)) \\ & +(\sigma I - A(U_{-,0}(0))) \cdot \partial_t U_-(0,0). \end{aligned} \tag{2.16}$$

In order to see the first order compatibility condition more clearly, let us diagonalize (2.16). Set

$$T_+ = (r_1(U_{+,0}(0)), r_2(U_{+,0}(0)), r_3(U_{+,0}(0))), \quad U_{+,0}(x) = T_+ V_{+,0}(x).$$

Then (2.16) can be expressed as:

$$M \cdot \begin{pmatrix} x''(0) \\ \partial_x V^2_{+,0}(0) \\ \partial_x V^3_{+,0}(0) \end{pmatrix} = k - (\sigma - \lambda_1(U_{+,0}(0)))^2 \cdot r_1(U_{+,0}(0)) \cdot \partial_x V^1_{+,0}(0) \tag{2.17}$$

where

$$M = ([U_0(0)], (\sigma - \lambda_2(U_{+,0}(0)))^2 \cdot r_2(U_{+,0}(0)), (\sigma - \lambda_3(U_{+,0}(0)))^2 \cdot r_3(U_{+,0}(0))),$$

$$k = -(\sigma I - A(U_{+,0}(0))) \cdot B(U_{+,0}(0)) + (\sigma I - A(U_{-,0}(0))) \cdot \partial_t U_-(0,0),$$

From the stability condition (A2),we know that the matrix M is invertible. Then, we have

$$\begin{pmatrix} x''(0) \\ \partial_x V^2_{+,0}(0) \\ \partial_x V^3_{+,0}(0) \end{pmatrix} = M^{-1} \cdot (k - (\sigma - \lambda_1(U_{+,0}(0)))^2 \cdot r_1(U_{+,0}(0)) \cdot \partial_x V^1_{+,0}(0)) \tag{2.18}$$

Thus, the second equation and the third equation of (2.18) are what we need here, one-order compatibility conditions, with $x''(0)$ determined from the first one.

Now, we can state the main theorem we want to prove in this paper.

Theorem 2.1: *Suppose that the initial data $U_{\pm,0}(x) \in C^1(\omega_\pm)$ satisfies the compatibility conditions (2.13) and (2.18), there exists $T > 0$ such that the problem (2.12) has unique solutions $(U_+, x(t))$, $U_+ \in C^1(\aleph_T^+), x(t) \in C^2[0,T]$, where $\aleph_T^+ = \aleph^+ \bigcap \{t < T\}$.*

Remark 2.2: As the transformation $\begin{cases} \tilde{x} = x - x(t) \\ \tilde{t} = t \end{cases}$ in §2.2 is invertible, we can get the local existence of 1-shock wave solutions to the original problem (2.8).

3 The existence of shock wave

This section is devoted to the proof of Theorem 2.1. For nonlinear problem (2.12), we first construct an approximate solution. Then we use iterations to construct a sequence of approximate solutions. By using a series of priori estimates and a compactness argument, we get the convergence of the sequence of approximate solutions. The limit of this sequence gives the solutions to problem (2.12). That also proves Theorem 2.1, or rather, the local existence of a shock wave solution of the original radiation hydrodynamic system (1.1).

3.1 Construction of the zero-th order approximate solution

Introduce the notations

$$L(U_+, x(t)) = \partial_t + (A(U_+) - x'(t)I)\partial_x \tag{3.1}$$

and

$$G(U_+, x'(t)) = x'(t)[U] - [F(U)]. \tag{3.2}$$

Obviously, problem (2.12) can be written as

$$\begin{cases} L(U_+, x(t))U_+ = B(U_+), \qquad (t, x > 0) \\ G(U_+, x'(t)) = 0, \qquad x = 0 \\ x(0) = 0 \\ U_+(0,x) = U_{+,0}(x). \end{cases} \tag{3.3}$$

Under the compatibility conditions (2.13) and (2.18), we try to construct approximate solutions $(U_+^0, x^0(t))$ of problem (3.3), such that $U_+^0 \in C^1(\aleph^+), x^0(t) \in C^2[0,T]$, and when $t = 0$, they satisfy

$$\begin{cases} L(U_+^0, x^0(t))U_+^0 = B(U_+^0) \\ d_t^k G(U_+^0, d_t x^0(t)) = 0, \qquad (k = 0,1) \\ x^0(0) = 0 \\ U_+^0(0,x) = U_{+,0}(x). \end{cases} \tag{3.4}$$

Set $m^k(x) = \partial_t^k U_+^0(0,x)$ for $k \in \{0,1\}$. The initial condition in (3.3) implies that $m^0(x) = U_{+,0}(x) \in C^1(\omega^+)$, while from the equation in (3.3), we deduce $m^1(x) \in C^0(\omega^+)$.

Let us construct the approximate solutions U_+^0 of (3.3) by the following lemma. The proof of Lemma 3.1 is omitted since it is the similar as that in [18] (Lemma 3.1) by Wang, Ya-Guang.

Lemma 3.1: *Given functions $m^0 \in C^1(\omega)$, and set*

$$m^1(x) = -(A(m^0(x)) - \sigma I)d_x m^0(x) + B(m^0(x)) \tag{3.5}$$

it can be obtained in (3.3), then there are functions $U_+^0 \in C^1(\aleph^+)$ such that

$$U_+^0(0,x) = m^0(x), \quad \partial_t U_+^0(0,x) = m^1(x) \tag{3.6}$$

and when $t = 0$,

$$\partial_t U_+^0 + (A(U_+^0) - \sigma I)\partial_x U_+^0 = B(U_+^0). \tag{3.7}$$

Set $b = x''(0)$, which can be got from (2.18), then define the approximate solution $x^0(t)) \in C^2[0,T]$ of (3.3) by the following lemma.

Lemma 3.2: *Let $U_+^0 \in C^1(\aleph^+)$ be the approximate solution of (3.3) given as above. Then there is a $x^0(t) \in C^2[0,T]$ such that*

$$\begin{cases} d_t^k G(U_+^0, d_t x^0(t))|_{t=0} = 0, \qquad (k = 0,1) \\ x^0(0) = 0, \quad d_t x^0(0) = \sigma, \quad d_t^0 x^0(0) = b \end{cases} \tag{3.8}$$

The proof of Lemma 3.2 is similar as that in [18] (Lemma 3.2).

Summing up, we have the following

Proposition 3.1: *Under the compatibility conditions (2.13) and (2.18), there are approximate solutions $(U_+^0, x^0(t))$ to the problem (3.3), such that $U_+^0 \in C^1(\aleph^+), x^0(t) \in C^2[0,T]$, and when $t = 0$, they satisfy (3.4),*

$$\begin{cases} L(U_+^0, x^0(t))U_+^0 = B(U_+^0) \\ d_t^k G(U_+^0, d_t x^0(t)) = 0, \qquad (k = 0,1) \\ x^0(0) = 0 \\ U_+^0(0,x) = U_{+,0}(x). \end{cases}$$

3.2 Construction of a sequence of approximate solutions

From the definition (3.2) of $G(U_+, x'(t))$, it is easy to see that the Fréchet derivative of G with respect to its arguments at $(V_+, \varphi'(t))$ is

$$G'_{(U_+,x'(t))}(V_+, \varphi'(t)) = (x'(t)I - A(U_+))V_+ + [U]\varphi'(t). \tag{3.9}$$

Given approximate solutions $(U_+^0, x^0(t))$ by Proposition 3.1, we solve the problem (2.12) by the iteration scheme

$$\begin{cases} L(U_+^\gamma, x^\gamma(t))U_+^{\gamma+1} = B(U_+^\gamma) \\ G'_{(U_+^\gamma, d_t x^\gamma(t))}(U_+^{\gamma+1}, d_t x^{\gamma+1}(t)) = -G(U_+^\gamma, d_t x^\gamma(t)) + G'_{(U_+^\gamma, d_t x^\gamma(t))}(U_+^\gamma, d_t x^\gamma(t)) \\ x^{\gamma+1}(0) = 0 \\ U_+^{\gamma+1}(0,x) = U_{+,0}(x) \end{cases} \tag{3.10}$$

that is, we use the usual Picard iteration for the equation, and the Newton iteration for the boundary condition. The zero-th and first order compatibility conditions (2.13) and (2.18) in (3.10) are satisfied.

To study the problem (3.10), let us first consider the linear problem (3.11) first. Then, we have following Proposition 3.2 towards it, thus, the local existence of the solutions to problem (3.10) and priori estimates are given, preparing well for the proof of the convergence of the sequence of approximate solutions in §3.4, and the proof of Proposition 3.2 will be given in §3.3.

$$\begin{cases} L(U_+, x(t))V_+ = B(U_+) + f \\ G'_{(U_+,x'(t))}(V_+, \varphi'(t)) = g(t) \\ \varphi(0) = 0 \\ V_+(0,x) = U_{+,0}(x) \end{cases} \tag{3.11}$$

where $U_+ \in C^1(\aleph_T^+), x(t) \in C^2(0,T)$, while $f \in C^1(\aleph_T^+)$ and $g \in C^1[0,T]$ satisfy the compatibility conditions of (3.11) up to order one.

To alleviate the burden of notations, in the remainder of this paper, setting $\omega_s^+ = \aleph^+ \bigcap \{t = s\}$, we use $\|u(t)\|$ and $\|u(t)\|_1 = \|u(t)\| + \|\nabla u(t)\|$ to denote the $L^\infty(\omega_t^+)$ and $W^{1,\infty}(\omega_t^+)$ norms, respectively, of $u(t,\cdot)$. Analogously, we use $\|u\|_t$ and $\|u\|_{1,t}$ to denote the $L^\infty(\aleph_t^+)$ and $W^{1,\infty}(\aleph_t^+)$ norms, respectively, of $u(\cdot)$. For any $\phi \in L^\infty[0,T]$, the norm $\|\phi\|_{L^\infty[0,t]}$ is also denoted by $\|\phi\|_t$, for any $t \in (0,T]$.

For the problem (3.11), we have the following results, the proof of which will be given in the next subsection.

Proposition 3.2:

(1) *Suppose that $f \in C^0(\aleph_T), g \in C^0[0,T]$ satisfy the zero-th order compatibility condition of the problem (3.11). Then there exist unique weak solutions $V_+ \in C^0(\aleph_T^+), \varphi \in C^1[0,T]$. to the problem (3.11). Moreover, there is a constant $C > 0$ such that*

$$|d_t\varphi(t)| + \|V_+(t)\| \le Ce^{CMt}(\|g\|_t + \|U_{+,0}\| + \int_0^t \|f(s) + B(U_+(s))\|ds) \tag{3.12}$$

for any $t \in (0,T]$, where M is a constant.

(2) *If* $(f,g)\in C^1$ *and satisfy the first order compatibility condition of the problem (3.11), then the unique solutions* $V_+\in C^1(\aleph_T^+), \varphi\in C^2[0,T]$. *Moreover, we have the estimate*

$$
\begin{aligned}
&|d_t^2\varphi(t)|+\|\nabla_{(t,x)}V_+(t)\|\\
&\le C\exp(CMte^{CMt})(\|d_tg\|_t+\|f(0)\|+\|B(U_+(0))\|+\|d_xU_{+,0}\|+M(\|g\|_t+\|U_{+,0}\|)\\
&\quad+\textstyle\int_0^t(\|\partial_tf(s)\|+M\|f(s)\|+\|B(U_+(s))\|)ds)
\end{aligned}
\tag{3.13}
$$

3.3 Proof of Proposition 3.2

In this subsection, we will give the proof of proposition 3.2. At first, let us diagnoalize the problem (3.11). Set

$$T_+=(r_1(U_+),r_2(U_+),r_3(U_+))$$

and

$$(T_+)^{-1}=(l_1(U_+),l_2(U_+),l_3(U_+))^T,$$

with $\{r_i,l_i\}_{i=1}^3$ given before. Set

$$V_+=T_+\tilde V_+ \tag{3.14}$$

and

$$\tilde L(U_+,x(t))=\partial_t+(\Lambda(U_+)-x'(t)I)\partial_x \tag{3.15}$$

with

$$\Lambda(U)=\mathrm{diag}[\lambda_1(U),\lambda_2(U),\lambda_3(U)] \tag{3.16}$$

being a diagonal matrix with eigenvalues as its entries. By making use of the fact

$$(\partial T_+)(T_+)^{-1}=-T_+\partial(T_+)^{-1},$$

it is easy to see that the problem (3.11) is equivalent to

$$
\begin{cases}
\tilde L(U_+,x(t))\tilde V_+=(T_+)^{-1}(B(U_+)+f)+(\tilde L(U_+,x(t))(T_+)^{-1})T_+\tilde V_+\\
\tilde G'_{(U_+,x'(t))}(\tilde V_+,\varphi'(t))=g(t)\\
\varphi(0)=0\\
\tilde V_+(0,x)=\tilde V_{+,0}(x)=(T_+)^{-1}U_{+,0}(x)
\end{cases}
\tag{3.17}
$$

with

$$\tilde G'_{(U_+,x'(t))}(\tilde V_+,\varphi'(t))=\sum_{i=1}^3(x'(t)-\lambda_i(U_+))r_i(U_+)\tilde V_++(U_+-U_-)\varphi'(t). \tag{3.18}$$

To study the problem (3.17), let us first consider the diagonal problem

$$
\begin{cases}
\tilde L(U_+,x(t))V_+=B(U_+)+f\\
\tilde G'_{(U_+,x'(t))}(V_+,\varphi'(t))=g(t)\\
\varphi(0)=0\\
V_+(0,x)=V_{+,0}(x)
\end{cases}
\tag{3.19}
$$

where $f \in C^1(\aleph_T^+)$, and $g \in C^1[0,T]$ satisfy the compatibility conditions of (3.19) up to order one. We decompose V_+ into two parts,

$$V_{+,I} = V_{+,1} \qquad \text{and} \qquad V_{+,II} = (V_{+,2}, V_{+,3})^T. \tag{3.20}$$

The same decompositions of $(f, B, V_{+,0})$ as above are also denoted by $f_I, f_{II}; B_I, B_{II}$ and $V_{+,0}^I, V_{+,0}^{II}$. From the Lax entropy condition, we know that

$$\lambda_i(U_+) - x'(t) \begin{cases} < 0, & i = 1 \\ > 0, & i = 2, 3 \end{cases} \tag{3.21}$$

which implies that (3.19) is an initial value problem for the component $V_{+,I}$, and a mixed problem for the component $V_{+,II}$.

Therefore, we immediately obtain the following lemma for $V_{+,I}$.

lemma 3.3:

(1) *For any bounded $U_+ \in C^1(\aleph_T^+)$ and $x(t) \in C^2[0,T], f_I \in C^0(\aleph_T^+)$ and $V_{+,0} \in C^0(\omega^+)$, there are unique weak solution $V_{+,I} \in C^0(\aleph_T^+)$ to the I-part of the problem (3.19). Moreover, for any $t \in (0,T]$,*

$$\|V_{+,I}(t)\| \leq \|V_{+,0}^I\| + \int_0^t \|f_I(s) + B_I(U_+(s))\| ds. \tag{3.22}$$

(2) *There are constants $C, M > 0$,*

$$\omega(\delta, t; V_{+,I}) \leq Ce^{CMt}\omega(\delta; V_{+,0}^I) + \delta\|f_I + B_I(U_+)\|_t + \int_0^t Ce^{CM(t-s)}\omega(\delta, s; f_I + B_I(U_+))ds \tag{3.23}$$

where

$$\omega(\delta, t; U) = \sup|U(s,x) - U(s', x')| \tag{3.24}$$

denotes the modulus of continuity of U with supremum taken over (s,x) and (s',x') in $\aleph_t^+$ such that $|(s,x) - (s',x')| \leq \delta$.

Proof: The proof is simple, the main idea is to get the value of $V_{+,I}$ by integrating through the characteristic curve. We will mainly discuss about $V_{+,II}$ then. Through the proof of estimation of $V_{+,II}$, we could get the results given in Lemma 3.3 analogously.¶

For the II-part of the problem (3.19)

$$\begin{cases} \partial_t V_{+,II} + \Theta_{II}(U_+, x(t))\partial_x V_{+,II} = f_{II} + B_{II} \\ M \cdot (\varphi'(t), V_{+,2}, V_{+,3})^T = g(t) + (\lambda_1(U_+) - x'(t))r_1(U_+)V_{+,1} \\ \varphi(0) = 0 \\ V_{+,II}(0,x) = V_{+,0}^{II}(x) \end{cases} \tag{3.25}$$

where $V_{+,1} \in C^0(\aleph^+)$ is given by Lemma 3.3,

$$\Theta_{II}(U_+, x(t)) = (\Lambda_{II}(U_+) - x'(t)I) = \text{diag}[\lambda_2(U_+) - x'(t), \lambda_2(U_+) - x'(t)]$$

is diagonal matrix with positive entries, and the matrix

$$M = ((U_+ - U_-), (x'(t) - \lambda_2(U_+))r_2(U_+), (x'(t) - \lambda_3(U_+))r_3(U_+))$$

is invertible from the stability condition (A2).

Without loss of generality, let us investigate the component $V_{+,2}$ in the problem (**??**). Obviously, we know that $V_{+,2}$ satisfies

$$\begin{cases} \partial_t V_{+,2} + (\lambda_2(U_+) - x'(t))\partial_x V_{+,2} = f_2 + B_2 \\ V_{+,2}(t,0) = a_2(t) \\ V_{+,2}(0,x) = V^2_{+,0}(x) \end{cases} \tag{3.26}$$

where $a_2(t)$ is the second component of the vector $(M)^{-1}\cdot(g(t)+(\lambda_1(U_+)-x'(t))r_1(U_+)V_{+,1})$, and the compatibility conditions of (3.26) up to order one are valid.

For the problem (3.26), by integrating along characteristic curves, we obtain the following

lemma 3.4:

(1) *For any $V^2_{+,0} \in C^0(\omega^+)$, and $f_2 \in C^0(\aleph^+_T)$, there is a unique weak solution $V_{+,2} \in C^0(\aleph^+_T)$ to the problem (3.26). Moreover, for any $t \in (0,T]$,*

$$\|V_{+,2}(t)\| \le \|a_2\|_t + \|V^2_{+,0}\| + \int_0^t \|f_2(s) + B_2(U_+(s))\| ds. \tag{3.27}$$

(2)

$$\omega(\delta,t;V_{+,2}) \le Ce^{CMt}(\omega(\delta,t;a_2) + \omega(\delta,V^2_{+,0}) + \delta\|f_2 + B_2\|_t + \int_0^t \omega(\delta,s;f_2 + B_2)ds) \tag{3.28}$$

Proof: Let $s \to (s,\Upsilon(s;t,x))$ be the characteristic curve of (3.26) through (t,x) with $\Upsilon(s;t,x)$ being the solution of the problem

$$\begin{cases} d_s\Upsilon(s;t,x) = \lambda_2(U_+(s,\Upsilon(s;t,x))) - x'(s) \\ \Upsilon(t;t,x) = x. \end{cases} \tag{3.29}$$

Let $s_1(t,x)$ be the root of

$$\Upsilon(s_1(t,x);t,x) = 0 \tag{3.30}$$

and

$$\tilde{\aleph}^+_T = \{(s,t,x)|\max(0,s_1(t,x)) \le s \le x \quad \text{for} \quad (t,x) \in \aleph^+_T\}.$$

From the theory of ordinary differential equations, we have $\Upsilon(s;t,x) \in C^1(\tilde{\aleph}^+_T)$. For $(t,x) \in \aleph^+_T$, we have two cases:

Case (1): $s_1(t,x) < 0$. In this case, (3.26) is a Cauchy problem for $V_{+,2}$, and for its solution we have the explicit formula

$$V_{+,2}(t,x) = V^2_{+,0}(\Upsilon(0;t,x)) + \int_0^t f_2(s,\Upsilon(s;t,x))ds + B_2(U_+(s,\Upsilon(s;t,x)))ds. \tag{3.31}$$

Case (2): $s_1(t,x) \ge 0$. The solution of problem (3.26) can be expressed as

$$V_{+,2}(t,x) = a_2(s_1(t,x)) + \int_{s_1(t,x)}^t f_2(s,\Upsilon(s;t,x))ds + B_2(U_+(s,\Upsilon(s;t,x)))ds. \tag{3.32}$$

From (3.31) and (3.32), we immediately deduce the estimate (3.27).

Next, we consider the estimate (3.28). For any $\delta > 0, t \in (0, T]$ and $(t_i, x_i) \in \aleph_t^+ \quad (i = 1, 2)$ with $|(t_1, x_1) - (t_2, x_2)| < \delta$, we divide the estimate of $V_{+,2}(t_1, x_1) - V_{+,2}(t_2, x_2)$ into three cases:

Case (α): $s_1(t_i, x_i) < 0 \quad (i = 1, 2)$. As above, (3.26) is a Cauchy problem for $V_{+,2}(t_i, x_i)$. We have

$$\begin{aligned} & |V_{+,2}(t_1, x_1) - V_{+,2}(t_2, x_2)| \\ & \quad \leq Ce^{CMt}\omega(\delta, V_{+,0}^2) + \delta\|f_2 + B_2\|_t + \textstyle\int_0^t Ce^{CM(t-s)}\omega(\delta, s; f_2 + B_2(U_+))ds. \end{aligned} \tag{3.33}$$

Case (β): $s_1(t_i, x_i) \geq 0 \quad (i = 1, 2)$. From (3.32), we have

$$\begin{aligned} & V_{+,2}(t_1, x_1) - V_{+,2}(t_2, x_2) \\ & \quad = a_2(s_1(t_1, x_1)) - a_2(s_1(t_2, x_2)) \\ & \quad\quad + \textstyle\int_{s_1(t_1,x_1)}^{t_1} (f_2(s, \Upsilon(s; t_1, x_1))ds + B_2(U_+(s, \Upsilon(s; t_1, x_1)))ds) \\ & \quad\quad - \textstyle\int_{s_1(t_2,x_2)}^{t_2} (f_2(s, \Upsilon(s; t_2, x_2))ds + B_2(U_+(s, \Upsilon(s; t_2, x_2)))ds). \end{aligned} \tag{3.34}$$

The definition (3.30) of $s_1(t, x)$ implies

$$0 = d_s\Upsilon(s_1(t, x); t, x) \cdot \partial s_1(t, x) + \partial\Upsilon(s_1(t, x); t, x)$$

Thus,

$$\begin{aligned} \partial s_1(t, x) & = -(d_s\Upsilon(s_1(t, x); t, x))^{-1} \cdot (\partial\Upsilon)(s_1(t, x); t, x) \\ & = (x'(t) - \lambda_2(U_+))^{-1} \cdot (\partial\Upsilon)(s_1(t, x); t, x) \end{aligned} \tag{3.35}$$

for $\partial = \partial_t$ or $\partial = \partial_x$. And we have the estimate of Υ,

$$|\partial_{(t,x)}\Upsilon(s; t, x)| \leq Ce^{CM(t-s)}. \tag{3.36}$$

The proof of (3.36) is given as follows, as

$$\begin{cases} \partial_s\Upsilon(s; t, x) = \lambda(U(s, \Upsilon(s; t, x))) \\ \Upsilon(t; t, x) = x, \end{cases}$$

so, we obtain

$$\Upsilon(s; t, x) = x + \int_s^t \lambda(U(\tau, \Upsilon(\tau; t, x)))d\tau.$$

1. $\partial_x\Upsilon(s; t, x) = 1 + \int_s^t \nabla_u\lambda \cdot U_x \cdot \partial_x\Upsilon(\tau; t, x)d\tau$

$$|\partial_x\Upsilon(s; t, x)| \leq 1 + C_1M \int_s^t |\partial_x\Upsilon(\tau; t, x)|d\tau$$

then, by applying Gronwall's inequality, we have

$$|\partial_x\Upsilon(s; t, x)| \leq e^{C_1M\int_s^t d\tau} = e^{C_1M(t-s)}.$$

2. $\partial_t\Upsilon(s; t, x) = 1 + \int_s^t \nabla_u\lambda \cdot U_t \cdot \partial_t\Upsilon(\tau; t, x)d\tau + \lambda(U(\tau, \Upsilon(\tau; t, x)))$

$$|\partial_t\Upsilon(s; t, x)| \leq C_2 + C_1M \int_s^t |\partial_t\Upsilon(\tau; t, x)|d\tau$$

similarly, by using Gronwall's inequality, we get

$$|\partial_t \Upsilon(s;t,x)| \le C_2 e^{C_1 M \int_s^t d\tau} = C_2 e^{C_1 M(t-s)}.$$

Taking $C = \max\{C_1, C_2\}$, we obtain the estimate (3.36),

$$|\partial_{(t,x)} \Upsilon(s;t,x)| \le C e^{CM(t-s)}.$$

Applying (3.36) in (3.35), it follows

$$|\partial_{(t,x)} s_1(t,x)| \le C e^{CMt} \qquad \text{when} \quad s_1(t,x) \ge 0 \tag{3.37}$$

with another constant $C > 0$. Employing (3.37) for (3.39), it is easy to obtain

$$\begin{aligned} &|V_{+,2}(t_1,x_1) - V_{+,2}(t_2,x_2)| \\ &\le C e^{CMt}(\omega(\delta,t;a_2) + \delta \|f_2 + B_2\|_t + \textstyle\int_0^t \omega(\delta,s;f_2+B_2)ds). \end{aligned} \tag{3.38}$$

Case (γ): $s_1(t_1,x_1) \ge 0$ and $s_1(t_2,x_2) < 0$. It is same for $s_1(t_1,x_1) < 0$ and $s_1(t_2,x_2) \ge 0$. From (3.31) and (3.32), we obtain

$$\begin{aligned} &V_{+,2}(t_1,x_1) - V_{+,2}(t_2,x_2) \\ &= a_2(s_1(t_1,x_1)) - V_{+,0}^2(t_2,x_2) \\ &+ \textstyle\int_{s_1(t_1,x_1)}^{t_1} (f_2(s,\Upsilon(s;t_1,x_1))ds + B_2(U_+(s,\Upsilon(s;t_1,x_1)))ds) \\ &- \textstyle\int_0^{t_2} (f_2(s,\Upsilon(s;t_2,x_2))ds + B_2(U_+(s,\Upsilon(s;t_2,x_2)))ds). \end{aligned} \tag{3.39}$$

Obviously, when $(t_i,x_i) \in \aleph_t^+$, and $|(t_1,x_1) - (t_2,x_2)| < \delta$, using (3.36) and (3.37), we have

$$0 \le s_1(t_1,x_1) \le s_1(t_1,x_1) - s_1(t_2,x_2) \le C e^{CMt}\delta \tag{3.40}$$

and

$$0 \le \Upsilon(0;t_2,x_2) \le \Upsilon(0;t_2,x_2) - \Upsilon(0;t_1,x_1) \le C e^{CMt}\delta. \tag{3.41}$$

Applying (3.40) and (3.41) in (3.39), it follows

$$\begin{aligned} &|V_{+,2}(t_1,x_1) - V_{+,2}(t_2,x_2| \\ &\le C e^{CMt}(\omega(\delta,t;a_2) + \omega(\delta,V_{+,0}^2) + \delta \|f_2 + B_2\|_t + \textstyle\int_0^t \omega(\delta,s;f_2+B_2)ds). \end{aligned} \tag{3.42}$$

Summing the above three cases up, it concludes the result (3.28).¶

From Lemmas 3.3 with 3.4 together it follows

lemma 3.5:

(1) *For any $f \in C^0(\aleph_T^+)$ and $V_{+,0} \in C^0(\omega^+)$ satisfying the zero-th order compatibility condition of (3.19). there are unique weak solutions $V_+ \in C^0(\aleph_T^+)$ and $\varphi \in C^1[0,T]$ to the problem (3.19). Moreover, there is a constant $C > 0$ such that, for any $t \in (0,T]$,*

$$|d_t\varphi(t)| + \|V_+(t)\| \le C(\|g\|_t + \|V_{+,0}\| + \int_0^t \|f(s) + B(U_+(s))\|ds). \tag{3.43}$$

(2) *If there is a constant $M > 0$, then*

$$\begin{aligned}&\omega(\delta, t; d_t\varphi(t)) + \omega(\delta, t; V_+)\\&\quad\leq Ce^{CMt}(\omega(\delta, t; g) + \omega(\delta, V_{+,0}) + \delta\|f + B\|_t + \textstyle\int_0^t \omega(\delta, s; f + B)ds)\end{aligned} \tag{3.44}$$

We can continue our research on Lemma 3.5, and give Lemma 3.6 without proving. (refers to [3])

lemma 3.6: *With the same conditions as above, if $g \in C^1[0, T]$ and $f \in C^0(\aleph_T^+)$ have the form*

$$f_i = \rho_i(\partial_t\sigma_i + (\lambda_i(U_+) - x'(t))\partial_x\sigma_i) \tag{3.45}$$

with $(\rho_i, \sigma_i) \in C^1$ for any $i \in \{1, 2, 3\}$, then the solutions of the problem (3.19) $V_+ \in C^1(\aleph_T^+), \varphi \in C^2[0, T]$.

Now we can give the proof of Proposition 3.2 as we have the result Lemma 3.6.

The proof of Proposition 3.2: Assertion (1): Let us turn to the study of the problem (3.17) before consider about (3.11). The diagonal problem (3.17) can be solved by the iteration scheme

$$\begin{cases}\tilde{L}(U_+, x(t))\tilde{V}_+^{\gamma+1} = (T_+)^{-1}(B(U_+) + f) + (\tilde{L}(U_+, x(t))(T_+)^{-1})T_+\tilde{V}_+^{\gamma}\\ \tilde{G}'_{(U_+, x'(t))}(\tilde{V}_+^{\gamma+1}, \varphi'(t)^{\gamma+1}) = g(t)\\ \varphi(0)^{\gamma+1} = 0\\ \tilde{V}_+^{\gamma+1}(0, x) = \tilde{V}_{+,0}(x)\end{cases} \tag{3.46}$$

with the first approximate solution $\tilde{V}_+^0 \in C^1(\aleph_T^+), \varphi^0 \in C^2[0, T]$ constructed in a way similar to Proposition 3.1. Under the assumption of Proposition 3.2(1), by employing the estimate (3.43) for the problem (3.46), there is a constant $C > 0$ such that, for any $\gamma \geq 0$ and $t \in (0, T]$,

$$\begin{aligned}&|d_t\varphi^{\gamma+1}(t)| + \|\tilde{V}_+^{\gamma+1}(t)\|\\&\quad\leq C(\|g\|_t + \|\tilde{V}_{+,0}\| + \textstyle\int_0^t(\|f(s) + B(U_+(s))\| + M\|\tilde{V}_+^{\gamma}(s)\|)ds).\end{aligned} \tag{3.47}$$

By induction on γ for (3.47), it follows that $\tilde{V}_+^{\gamma} \in C^0(\aleph_T^+), \varphi^{\gamma} \in C^1[0, T]$. By using Gronwall's inequality, we have

$$|d_t\varphi^{\gamma}(t)| + \|\tilde{V}_+^{\gamma}(t)\| \leq Ce^{CMt}(\|g\|_t + \|\tilde{V}_{+,0}\| + \int_0^t \|f(s) + B(U_+(s))\|ds) \tag{3.48}$$

for any $\gamma > 0, t \in (0, T]$. Moreover, by employing (3.43) for the problem of $(\tilde{V}_+^{\gamma+1} - \tilde{V}_+^{\gamma}, \varphi^{\gamma+1} - \varphi^{\gamma})$

$$\begin{cases}\tilde{L}(U_+, x(t))(\tilde{V}_+^{\gamma+1} - \tilde{V}_+^{\gamma}) = (\tilde{L}(U_+, x(t))(T_+)^{-1})T_+(\tilde{V}_+^{\gamma} - \tilde{V}_+^{\gamma-1})\\ \tilde{G}'_{(U_+, x'(t))}(\tilde{V}_+^{\gamma+1} - \tilde{V}_+^{\gamma}, \varphi'(t)^{\gamma+1} - \varphi'(t)^{\gamma}) = 0\\ \varphi^{\gamma+1} - \varphi^{\gamma}|_{t=0} = \tilde{V}_+^{\gamma+1} - \tilde{V}_+^{\gamma}|_{t=0} = 0\end{cases} \tag{3.49}$$

we obtain

$$|d_t(\varphi^{\gamma+1} - \varphi^\gamma)(t)| + \|(\tilde{V}_+^{\gamma+1} - \tilde{V}_+^\gamma)(t)\| \leq CM \int_0^t \|\tilde{V}_+^\gamma - \tilde{V}_+^{\gamma-1}(s)\| ds$$

which implies

$$|d_t(\varphi^{\gamma+1} - \varphi^\gamma)(t)| + \|(\tilde{V}_+^{\gamma+1} - \tilde{V}_+^\gamma)(t)\| \leq \frac{(CMt)^\gamma}{\gamma!} \|\tilde{V}_+^0\|_T \tag{3.50}$$

by induction on γ. From (3.48) and (3.50), and the uniqueness of weak solutions $(\tilde{V}_+, d_t\varphi)$ in L^∞ for the problem (3.17), which is a simple consequence from the estimate (3.43) in (3.17) with $(f, g, U_{+,0}) = 0$, we immediately deduce that the sequences $\tilde{V}_+^\gamma$ and φ^γ converge in $C^0(\aleph_T^+)$ and $C^1[0,T]$, the limits $\tilde{V}_+ \in C^0(\aleph_T^+)$ and $\varphi \in C^1[0,T]$ are the unique weak solutions of the problem (3.17), and they satisfy the estimate

$$|d_t\varphi(t)| + \|\tilde{V}_+(t)\| \leq Ce^{CMt}(\|g\|_t + \|\tilde{V}_{+,0}\| + \int_0^t \|f(s) + B(U_+(s))\| ds$$

for any $t \in (0,T]$.

Assertion(2): The righthand side of the iteration equation (3.46) is of the form (3.45). By applying Lemma 3.6 in (3.46) we conclude that $\tilde{V}_+^\gamma \in C^1(\aleph_T^+)$ and $\varphi^\gamma \in C^2[0,T]$. As a lemma in [5] resembling Lemma 3.5(2) applied to $\nabla\tilde{V}_+^\gamma$ and $d_t^2\varphi^\gamma$ shows that, the families $\{\nabla\tilde{V}_+^\gamma, d_t^2\varphi^\gamma\}_{\gamma\in I\!N}$ are equicontinuous. Therefore $\tilde{V}_+^\gamma \to \tilde{V}_+$ in $C^1(\aleph_T^+)$ and $\varphi^\gamma \to \varphi$ in $C^2[0,T]$ as $\gamma \to \infty$. Going back to the problem (3.11), it follows that, the solutions of the problem (3.11) V_+ and φ belong to $C^1(\aleph_T^+)$ and $C^2[0,T]$ respectively.

To estimate ∇V_+ and $d_t^2\varphi$ by setting $J_+ = \partial_t V_+$ and $\phi = d_t\varphi$ and differentiating (3.11) with respect to t, we obtain that (J_+, ϕ) satisfy the problem

$$\begin{cases} L(U_+, x(t))J_+ = Q \\ G'_{(U_+, x'(t))}(J_+, \phi') = \kappa(t) \\ J_+(0,x) = J_{+,0}(x) \end{cases} \tag{3.51}$$

where

$$\begin{aligned} Q = \ & (\partial_t f + \nabla B(U_+) \cdot \partial_t U_+) - \nabla A(U_+)(\partial_t U_+, D \cdot (f + B(U_+) - J_+)) \\ & + x''(t) D \cdot (f + B(U_+) - J_+) \end{aligned} \tag{3.52}$$

with

$$D = (A(U_+) - x'(t)I)^{-1},$$

$$\kappa(t) = d_t g(t) - x''(t) I \cdot V_+ + \nabla A(U_+)(\partial_t U_+, V_+) - [\partial_t U]\phi \tag{3.53}$$

and

$$J_{+,0}(x) = f(0,x) + B(U_+(0,x)) - (A(U_+) - x'(t)I)|_{t=0} \cdot d_x U_{+,0}(x). \tag{3.54}$$

Applying the estimate (3.12) in the problem (3.51), it follows

$$|d_t\phi(t)| + \|J_+(t)\| \leq Ce^{CMt}(\|\kappa\|_t + \|J_{+,0}\| + \int_0^t \|Q(s)\| ds). \tag{3.55}$$

From (3.52), we have

$$\|Q(s)\| \leq C(\|\partial_t f(s)\| + M\|f(s)\| + M\|B(s)\| + M\|J_+\|). \tag{3.56}$$

Obviously, (3.53) gives rise to

$$\|\kappa(t)\|_t \leq \|d_t g\|_t + M(\|\phi\|_t + C\|V_+\|_t) \tag{3.57}$$

which implies

$$\|\kappa(t)\|_t \leq \|d_t g\|_t + CMe^{CMt}(\|g\|_t + \|U_{+,0}\| + \int_0^t \|f(s) + B(U_+(s))\|ds) \tag{3.58}$$

by using (3.12). Substituting (3.56) and (3.58) into (3.55), and using Gronwall's inequality, we obtain

$$\begin{aligned}
&|d_t\phi(t)| + \|J_+(t)\| \\
&\leq C\exp(CMte^{CMt})(\|d_t g\|_t + \|f(0,x)\| + \|B(U_+(0,x))\| + \|d_x U_{+,0}\| \\
&\quad +M(\|g\|_t + \|U_{+,0}\|) + \textstyle\int_0^t(\|\partial_t f(s)\| + M\|f(s)\| + \|B(U_+(s))\|)ds).
\end{aligned} \tag{3.59}$$

The estimate of $\partial_x V_+$ can be easily obtained from the equation in (3.11).¶

3.4 Convergence of the sequence of approximate solutions

We construct a sequence of approximate solutions in §3.2, and give the local existence and priori estimates of solutions to the iteration scheme (3.10). Then, in this subsection, we introduce Lemma 3.7, therefore, Theorem 2.1 is proved at last.

lemma 3.7: *Suppose that we have $|d_t x - d_t x^0| + \|U_+ - U_+^0\| \leq \eta < 1$ in problem (3.3), such that $|d_t x^\gamma - d_t x^0| + \|U_+^\gamma - U_+^0\| \leq \eta$ and $|d_t^2(x^\gamma - x^0)| + \|\nabla_{(t,x)}(U_+^\gamma - U_+^0)(t)\| \leq \delta$ to the iteration scheme (3.10), while $\delta + \eta = \epsilon < 1$. There exists T which is relate to δ, but independent of U_+, such that when $0 \leq t \leq T$. Then we have $U_+^\gamma \to U_+$ in $C^1(\aleph_T^+)$ and $x^\gamma(t) \to x(t)$ in $C^2[0,T]$ as $\gamma \to \infty$.*

Proof: Let us state the clue of our proof first. We will prove that $U_+^\gamma \in C^1(\aleph_T^+)$, and $x^\gamma \in C^2(0,T]$, then for the convergence of $U_+^\gamma \to U_+$ in $C^0(\aleph_T^+)$ with $x^\gamma(t) \to x(t)$ in $C^1[0,T]$ as $\gamma \to \infty$. So, we have the result that $U_+^\gamma \to U_+$ in $C^1(\aleph_T^+)$ and $x^\gamma(t) \to x(t)$ in $C^2[0,T]$ as $\gamma \to \infty$. The solutions to problem (3.3) $U_+ \in C^1(\aleph_T^+), x(t) \in C^2[0,T]$.

Set

$$V_+ = U_+ - U_+^0, \quad y = x - x^0, \quad V_+^\gamma = U_+^\gamma - U_+^0, \quad y^\gamma(t) = x^\gamma(t) - x^0(t), \tag{3.60}$$

it is easy to see that the problem (3.3)and iteration scheme (3.10) can be expressed as

$$\begin{cases}
\partial_t V_+ + \partial_t U_+^0 + (A(V_+ + U_+^0) - (y'(t) + d_t x^0)I)\partial_x(U_+^0 + V_+) = B(U_+^0 + V_+) \\
G(V_+, y'(t)) = 0, \qquad y = 0 \\
y(0) = 0 \\
V_+(0, y) = 0
\end{cases} \tag{3.61}$$

with

$$\begin{cases} \partial_t V_+^{\gamma+1} + (A(V_+ + U_+^0) - (y'(t) + d_t x^0)I)\partial_x V_+^{\gamma+1} = H(V_+^\gamma) \\ G'_{(V_+^\gamma, d_t y^\gamma)}(V_+^{\gamma+1}, d_t y^{\gamma+1}) = g^\gamma \\ y^{\gamma+1}(0) = 0 \\ V_+^{\gamma+1}(0, y) = 0 \end{cases} \tag{3.62}$$

where

$$H(V_+^\gamma) = B(U_+^0 + V_+^\gamma) - \partial_t U_+^0 - (A(V_+^\gamma + U_+^0) - (d_t y^\gamma(t) + d_t x^0)I) \cdot \partial_x U_+^0,$$

$$g^\gamma = -G(V_+^\gamma, d_t y^\gamma) + G'_{(V_+^\gamma, d_t y^\gamma)}(V_+^\gamma, d_t y^\gamma).$$

Employing (3.12) for problem (3.62), we obtain

$$\begin{aligned} |d_t y^{\gamma+1}(t)| + \|V_+^{\gamma+1}(t)\| &\le Ce^{CMt}(\textstyle\int_0^t \|H(V_+^\gamma)(s)\| ds + \|g^\gamma\|_t) \\ &\le Ce^{CMt}(\textstyle\int_0^t (C_1 + C_2(\|U_+^\gamma(s)\|)\|V_+^\gamma(s)\|) ds + \|g^\gamma\|_t). \end{aligned} \tag{3.63}$$

As the assumption that $|d_t y^\gamma(t)| + \|V_+^\gamma(t)\| \le \eta < 1$, then we would like to see that $|d_t y^{\gamma+1}(t)| + \|V_+^{\gamma(t)+1}\| \le \eta$. Obviously, $\|g^\gamma\| \le C\eta^2$, so from (3.63), we have

$$|d_t y^{\gamma+1}(t)| + \|V_+^{\gamma+1}(t)\| \le Ce^{CMt}(C_1 t + C_2(\eta)\eta t + C\eta^2).$$

When T is small enough, we can see that $|d_t y^{\gamma+1}(t)| + \|V_+^{\gamma+1}(t)\| \le \eta$.

By using (3.13), we have

$$\begin{aligned} &|d_t^2 y^{\gamma+1}| + \|\nabla_{(t,x)} V_+^{\gamma+1}(t)\| \\ &\le C\exp(CMte^{CMt})(\|d_t g^\gamma\|_t + \|H(V_+^\gamma(0,x))\| + M\|g\|_t + \textstyle\int_0^t (\|(\partial_t H(V_+^\gamma)(s)\| + M\|H(V^\gamma)(s)\|) ds) \\ &\le C\exp(CMte^{CMt})(M\|g^\gamma\|_{1,t} + M\textstyle\int_0^t \|H(V^\gamma)(s)\|_1 ds) \end{aligned}$$

and as $|d_t^2 y^\gamma| + \|\nabla_{(t,x)} V_+^\gamma(t)\| \le \delta$ such that $\delta + \eta = \epsilon < 1$. Thus, we have the following

$$|d_t^2 y^{\gamma+1}| + \|\nabla_{(t,x)} V_+^{\gamma+1}(t)\| \le C\exp(CMte^{CMt}(MC(\epsilon)\epsilon^2 + M(C_1 t + C_2(\epsilon)\epsilon t)).$$

When T is small enough, we obtain $|d_t^2 y^{\gamma+1}| + \|\nabla_{(t,x)} V_+^{\gamma+1}(t)\| \le \delta$. So, if K and T are small enough, we have

$$\|U_+^\gamma\|_{1,\aleph_T^+} + \|d_t x(t)\|_{1,[0,T]} \le K. \tag{3.64}$$

Hence, we have $U_+^\gamma \in C^1(\aleph_T^+)$ and $x^\gamma(t) \in C^2(0,T]$.

From (3.10), we know that $(U_+^{\gamma+1} - U_+^\gamma, x^{\gamma+1}(t) - x^\gamma(t))$ satisfy the problem

$$\begin{cases} L(U_+^\gamma, x^\gamma(t))(U_+^{\gamma+1} - U_+^\gamma) = M^\gamma \\ G'_{(U_+^\gamma, d_t x^\gamma(t))}(U_+^{\gamma+1} - U_+^\gamma, d_t x^{\gamma+1}(t) - d_t x^\gamma(t)) = \kappa^\gamma \\ (U_+^{\gamma+1} - U_+^\gamma)(0, x) = 0 \\ (x^{\gamma+1}(0) - x^\gamma(0)) = 0 \end{cases} \tag{3.65}$$

where

$$M^\gamma = L(U_+^{\gamma-1}, x^{\gamma-1}(t))U_+^\gamma - L(U_+^\gamma, x^\gamma(t))U_+^\gamma + B(U_+^\gamma) - B(U_+^{\gamma-1})$$

and

$$\kappa^\gamma = G'_{(U_+^{\gamma-1}, d_t x^{\gamma-1})}(U_+^\gamma, d_t x^\gamma) - G'_{(U_+^{\gamma-1}, d_t x^{\gamma-1})}(U_+^{\gamma-1}, d_t x^{\gamma-1}) - G(U_+^\gamma, d_t x^\gamma) + G(U_+^{\gamma-1}, d_t x^{\gamma-1}).$$

Setting $a^\gamma(t) = \|U_+^{\gamma+1} - U_+^\gamma\|_t + \|d_t(x^{\gamma+1}(t) - x^\gamma(t))\|_t < 1$, then we have

$$\|M^\gamma(s)\| \leq Ca^{\gamma-1}(s). \tag{3.66}$$

The function $G(U_+^\gamma, d_t x^\gamma)$ in κ^γ admits Taylor's expansion at $G(U_+^{\gamma-1}, d_t x^{\gamma-1})$. Thus, we get

$$\|\kappa^\gamma\| \leq C(a^{\gamma-1}(t))^2, \tag{3.67}$$

here, we should also use the result proved before that $U_+^\gamma \in C^1(\aleph_T^+)$ and $x^\gamma(t) \in C^2[0,T]$.

Then, by using the estimate (3.12) in the problem (3.65), we obtain

$$a^\gamma(t) \leq C((a^{\gamma-1}(t))^2 + \int_0^t a^{\gamma-1}(s)ds). \tag{3.68}$$

If T is small enough, it follows that $U_+^\gamma \to U_+$ in $C^0(\aleph_T^+)$ and $x^\gamma(t) \to x(t)$ in $C^1(0,T]$ as $\gamma \to \infty$, while $U_+ \in C^0(\aleph_T^+), x(t) \in C^1[0,T]$. As we known form (3.64), $U_+^\gamma \in C^1(\aleph_T^+)$ and $x^\gamma(t) \in C^2(0,T]$, Consequently, $U_+^\gamma \to U_+$ in $C^1(\aleph_T^+)$ and $x^\gamma(t) \to x(t)$ in $C^2(0,T]$ as $\gamma \to \infty$, that means the solutions to problem (3.3) $U_+ \in C^1(\aleph_T^+), x(t) \in C^2[0,T]$.¶

We get the conclusion of Lemma 3.7 form which we immediately obtain the conclusion of Theorem 2.1.

Acknowledgements. The authors would like to express their gratitude to their tutor Professor Ya-Guang Wang for his useful guidance and discussions.

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